

ON AN EFFICIENT METHOD FOR THE SIMULTANEOUS APPROXIMATION OF POLYNOMIAL MULTIPLE ROOTS

Miodrag S. Petković, Ljiljana D. Petković, Jovana Džunić

An iterative method in parallel mode for the simultaneous determination of multiple roots of algebraic polynomials is stated together with its single-step variant. These methods are more efficient compared to all simultaneous methods based on fixed point relations. To attain very high computational efficiency, a suitable correction resulting from LI-LIAO-CHENG's two-point fourth-order method of low computational complexity and Gauss-Seidel's approach are applied. Considerable increase of the convergence rate is obtained applying only ν additional polynomial evaluations per iteration, where ν is the number of distinct roots. A special emphasis is given to the convergence analysis and computational efficiency of the proposed methods. The presented convergence analysis shows that the R -order of convergence of the proposed single-step method is at least $2 + \tau_\nu$, where $\tau_\nu \in (4, 6)$ is the unique positive root of the polynomial $g_\nu(\tau) = \tau^\nu - 4^{\nu-1}\tau - 2^{2\nu-1}$. The convergence order of the corresponding total-step method is six. Computational aspects and some numerical examples are given to demonstrate high computational efficiency and very fast convergence of the proposed methods.

1. INTRODUCTION

The aim of this paper is to construct and study a new iterative method with a very high computational efficiency for the simultaneous determination of all multiple roots of a polynomial. Actually, the proposed method is ranked as the most efficient in the class of simultaneous methods for approximating polynomial multiple roots based on fixed point relations. The presented iterative formula relies on

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the fixed point relation of GARGANTINI's type [11]. A very high computational efficiency is attained by employing suitable corrections which enable very fast convergence (greater than six) with minimal additional computational cost. In fact, these corrections arise from LI-LIAO-CHENG's two-point root-solver [19] with optimal order of convergence in the sense of the KUNG-TRAUB conjecture [18]. More details about multi-point methods may be found, e.g., in [32]–[37].

There is a vast literature on the methods for finding polynomial roots, see, e.g., the books [16], [22], [23], [33], [40]. In this paper we concentrate on algorithms for the determination of multiple roots. There are various approaches in constructing these algorithms, which can be roughly divided into two classes: methods for finding one root and methods for determination of all roots of a given polynomial simultaneously. The first group includes, among others, quadratically convergent Newton-like methods, methods of Graeffe's type [20], [21], and one-point methods of higher order such as Laguerre's, Halley's, Chebyshev's, Ostrowski's methods (see [22]). These methods find one root at a time and they can be suitably modified to preserve the convergence rate in the presence of multiplicity of roots.

Iterative methods that approximate all polynomial roots simultaneously make the second group. The most frequently used procedures of this type are based on *root-relations* (called, also, *fixed point relations*) [22], [33], [40], and (companion) matrix iterations [16], [22, Ch. 6], [25], [45], [46], [47]. Note that the problem of finding multiple roots is closely connected with the computation of cluster of roots, see, e.g., [3], [15], [17], [26], [43], [47]. The computation of roots of polynomials with inexact coefficients was considered in [5], [34], [44], [45], [47], etc.

In this paper we restrict our attention to a class of simultaneous methods based on the root-relation (or the fixed point relation) of the form

$$\zeta_i = G_i(z_1, \dots, z_\nu, \zeta_1, \dots, \zeta_\nu) \quad (i = 1, \dots, \nu),$$

where $\zeta_1, \dots, \zeta_\nu$ are distinct (simple or multiple) roots of a given polynomial $f(z)$ and z_1, \dots, z_ν are their approximations, respectively. Some of arguments of the iteration functions G_i may appear once, several times or do not appear at all. Substituting the roots $\zeta_1, \dots, \zeta_\nu$ by new suitable approximations z_1^*, \dots, z_ν^* , permitting the choice $z_j^* = z_j$, presumably improved approximations

$$(1) \quad \hat{z}_i = G_i(z_1, \dots, z_\nu, z_1^*, \dots, z_\nu^*) \quad (i = 1, \dots, \nu)$$

are obtained. In this way, the iterative formula (1) defines a simultaneous method generating k (mutually dependent) sequences $\{z_i^{(k)}\}_{k=0,1,\dots}$ ($i = 1, \dots, \nu$).

The determination of all roots of a polynomial $f(z)$, *one-root-at-time-methods* can be performed in essentially two ways:

- (i) Determine all ν roots starting from ν distinct initial approximations $z_1^{(0)}, \dots, z_\nu^{(0)}$ and generate k independent sequences $\{z_i^{(k)}\}$, where $z_i^{(k+1)} = F(z_i^{(k)})$.
- (ii) Determine the desired roots serially where, after finding a root ζ_i , the corresponding linear factor $z - \zeta_i$ is removed from the polynomial and the process is employed again to determine a root of the "deflated" polynomial whose degree is now lowered (so-called method of successive *deflation*).

According to the above short study, we can observe the following:

Approximations obtained by the simultaneous methods (1) are coupled so that the improvement of any approximation improves the others. In this way the accuracy of all approximations is almost the same. This is one of the most important advantages of simultaneous methods. It is obvious that this property does not hold for one-root-at-time methods. Namely, since these methods are defined by ν independent formulae of the form $\hat{z}_i = F(z_i)$ (see the point (i) above), it may happen that some sequences, produced in this way, converge slowly, or even do not converge at all. Also, the approach of successive deflation (the point (ii) above) can cause that the polynomial obtained after divisions by insufficiently accurate linear factors $z - z^{(k)}$ may be “falsified” to an extent which makes the remaining approximate roots meaningless. The presence of rounding errors further aggravates this unpleasant situation. The mentioned advantages of simultaneous methods make these methods, in general, much more robust than one-root-at-time methods, which was extensively studied by SEMERDZIEV [39]. Finally, comparing these two classes of methods we also notice another advantage of simultaneous methods consisting of their parallel implementation on parallel computers, which is not feasible for one-root-at-time methods. For the displayed preferences, in what follows we will not consider one-root-at-time methods. Nevertheless, for demonstration, we give comparative results in Example 3 in Section 4.

The paper is organized as follows. In Section 2 we present a new iterative method in parallel and serial fashion for the simultaneous determination of polynomial multiple roots, starting from a suitable fixed-point relation and Li-Liao-Cheng’s two-point method for multiple roots. The lower bound of the R -order of convergence is determined in Section 3. Finally, Section 4 contains an analysis of computational efficiency which shows that the proposed simultaneous method is the most efficient among all methods based on fixed point relations. In addition, three numerical examples are given to demonstrate exceptional convergence speed of the proposed methods and to confirm the theoretical results.

2. MODIFIED GARGANTINI-LIKE METHOD

Let $f(z) = \prod_{j=1}^{\nu} (z - \zeta_j)^{\mu_j}$ be a monic polynomial of degree n with multiple real or complex roots $\zeta_1, \dots, \zeta_{\nu}$ of respective multiplicities μ_1, \dots, μ_{ν} ($\nu \leq n$), and let

$$(2) \quad u(z) = \frac{f(z)}{f'(z)} = \left[\frac{d}{dz} \log f(z) \right]^{-1} = \left(\sum_{j=1}^{\nu} \frac{\mu_j}{z - \zeta_j} \right)^{-1}.$$

To construct an iterative method for the simultaneous determination of polynomial multiple roots, we single out the term $z - \zeta_i$ from (2) and derive the following fixed point relation

$$(3) \quad \zeta_i = z - \frac{\mu_i}{\frac{1}{u(z)} - \sum_{\substack{j \in \mathbf{I}_\nu \\ j \neq i}} \frac{\mu_j}{z - \zeta_j}} \quad (i \in \mathbf{I}_\nu := \{1, \dots, \nu\}).$$

GARGANTINI has used this relation in [11] for the construction of iterative methods for the simultaneous inclusion of multiple roots of polynomials in complex circular arithmetic.

Let z_1, \dots, z_ν be distinct approximations to the roots $\zeta_1, \dots, \zeta_\nu$. Setting $z = z_i$ and substituting the roots ζ_j by some approximations z_j^* in the right-hand side of (3), one obtains the following iterative method

$$(4) \quad \hat{z}_i = z_i - \frac{\mu_i}{\frac{1}{u(z_i)} - \sum_{\substack{j \in \mathbf{I}_\nu \\ j \neq i}} \frac{\mu_j}{z_i - z_j^*}} \quad (i \in \mathbf{I}_\nu)$$

for the simultaneous determination of all multiple roots of the polynomial f . Here \hat{z}_i denotes a new approximation to the root ζ_i .

The choice $z_j^* = z_j$ in (4) gives the third-order method for finding multiple roots

$$(5) \quad \hat{z}_i = z_i - \frac{\mu_i}{\frac{1}{u(z_i)} - \sum_{\substack{j \in \mathbf{I}_\nu \\ j \neq i}} \frac{\mu_j}{z_i - z_j}} \quad (i \in \mathbf{I}_\nu).$$

Note that if all roots are simple, then (5) reduces to the Ehrlich-Aberth method, see [1], [7]. Furthermore, putting Schröder's approximations $z_j^* = z_j - \mu_j u(z_j)$ in (4), the following accelerated method of the fourth order is obtained (see [24]),

$$(6) \quad \hat{z}_i = z_i - \frac{\mu_i}{\frac{1}{u(z_i)} - \sum_{\substack{j \in \mathbf{I}_\nu \\ j \neq i}} \frac{\mu_j}{z_i - z_j + \mu_j u(z_j)}} \quad (i \in \mathbf{I}_\nu).$$

Note that the iterative method (6) reduces to NOUREIN's method [27] in the case of simple roots.

In what follows we will write for simplicity

$$\sum_{j < i} \quad \text{and} \quad \sum_{j > i} \quad \text{instead of} \quad \sum_{\substack{j \in \mathbf{I}_\nu \\ j < i}} \quad \text{and} \quad \sum_{\substack{j \in \mathbf{I}_\nu \\ j > i}}.$$

The iterative methods (5) and (6) will be referred to as the *total-step* methods or the methods in *parallel mode*. The convergence of this method can be accelerated by calculating the new approximation \hat{z}_i serially using the already calculated approximations $\hat{z}_1, \dots, \hat{z}_{i-1}$ as soon as they are available (the so-called Gauss-Seidel approach). The *single-step* variants of the total-step methods (5) and (6) are given

by

$$(7) \quad z_i^{(k+1)} = z_i^{(k)} - \frac{\mu_i}{\frac{1}{u(z_i^{(k)})} - \sum_{j<i} \frac{\mu_j}{z_i^{(k)} - z_j^{(k+1)}} - \sum_{j>i} \frac{\mu_j}{z_i^{(k)} - z_j^{(k)}}}$$

$(i \in \mathbf{I}_\nu, k = 0, 1, \dots)$

and

$$(8) \quad z_i^{(k+1)} = z_i^{(k)} - \frac{\mu_i}{\frac{1}{u(z_i^{(k)})} - \sum_{j<i} \frac{\mu_j}{z_i^{(k)} - z_j^{(k+1)}} - \sum_{j>i} \frac{\mu_j}{z_i^{(k)} - z_j^{(k)} + u(z_j^{(k)})}}$$

$(i \in \mathbf{I}_\nu, k = 0, 1, \dots)$.

According to the results for single-step methods for simple roots presented in [2] and [24], the lower bounds of the R -order of convergence of the single-step methods (7) and (8) can be derived:

(i) the R -order of convergence of the single-step method (7) is at least $2 + \tau_\nu$, where $\tau_\nu > 1$ is the unique positive root of the polynomial $G_{1,\nu}(\tau) = \tau^\nu - \tau - 2$;

(ii) the R -order of convergence of the single-step method (8) is at least $2 + \tau_\nu$, where $\tau_\nu > 2$ is the unique positive root of the polynomial $G_{2,\nu}(\tau) = \tau^\nu - 2^{\nu-1}\tau - 2^\nu$;

Regarding (3)–(6), it is evident that the better approximations z_j^* give the more accurate approximations \hat{z}_i . Indeed, if $z_j^* \rightarrow \zeta_j$ then the right-hand side of (4) tends to ζ_i . We apply this idea to construct a higher order method.

The iterative method (6) of the fourth order is obtained using Schröder's method $z_j^* = z_j - \mu_j u(z_j)$ of the second order. Further acceleration of the convergence speed can be obtained by using methods of higher order for finding a single multiple root. In this paper we use the following two-point method for solving nonlinear equations proposed by LI, LIAO and CHENG in [19]

$$(9) \quad z^* = L(z) := z - u(z) \cdot \frac{\beta + \gamma t(z)}{1 - \delta t(z)}, \quad t(z) = \frac{f'(z - \theta u(z))}{f'(z)},$$

where z^* is a new approximation,

$$\theta = \frac{2m}{m+2}, \quad \beta = -\frac{m^2}{2}, \quad \delta = \left(\frac{m+2}{m}\right)^m, \quad \gamma = \frac{m(m-2)}{2} \delta,$$

and m is the multiplicity of the wanted root ζ of a function f (not necessarily algebraic polynomial in general).

In the sequel, we substitute z by the approximation z_j of ζ_j and m by the corresponding multiplicity μ_j of ζ_j . The approximation z_j^* appearing in (4) is calculated by (9), that is,

$$z_j^* = L(z_j) := z_j - u_j \cdot \frac{\beta_j + \gamma_j t_j}{1 - \delta_j t_j},$$

where we put $u_j = u(z_j)$, $t_j = f'(z_j - \theta_j u_j)/f'(z_j)$ and

$$\theta_j = \frac{2\mu_j}{\mu_j + 2}, \quad \beta_j = -\frac{\mu_j^2}{2}, \quad \delta_j = \left(\frac{\mu_j + 2}{\mu_j}\right)^{\mu_j}, \quad \gamma_j = \frac{\mu_j(\mu_j - 2)}{2} \delta_j.$$

The order of convergence of the iterative method (9) is four, that is,

$$(10) \quad z_j^* - \zeta_j = L(z_j) - \zeta_j = c_j(z_j - \zeta_j)^4 + \mathcal{O}_M((z_j - \zeta_j)^5) = \mathcal{O}_M((z_j - \zeta_j)^4)$$

holds (for the proof, see [19]). Here \mathcal{O}_M is a symbol which points to the fact that two real or complex numbers w_1 and w_2 have moduli of the same order (that is, $|w_1| = \mathcal{O}(|w_2|)$, \mathcal{O} is the Landau symbol), written as $w_1 = \mathcal{O}_M(w_2)$.

REMARK 1. The quantity c_j , appearing in (10), has been determined in [19] and reads

$$c_j = \frac{\mu_j^3 + 2\mu_j^2 + 2\mu_j - 2}{3\mu_j^4(\mu_j + 1)^3} Q_{1,j}^3 - \frac{Q_{1,j}Q_{2,j}}{\mu_j(\mu_j + 1)^2(\mu_j + 2)} + \frac{\mu_j Q_{3,j}}{(\mu_j + 1)(\mu_j + 3)(\mu_j + 2)^3},$$

where $Q_{\lambda,j} = f^{(\mu_j + \lambda)}(\zeta_j)/f^{(\mu_j)}(\zeta_j)$. Although the asymptotic error constant c_j is given by a complicated expression, it is bounded in magnitude and this fact is quite sufficient for our purpose.

Now, using LI-LIAO-CHENG's approximations (9), we obtain from (4) a new method for the simultaneous approximation of all simple or multiple roots of a given polynomial,

$$(11) \quad z_i^{(k+1)} = z_i^{(k)} - \frac{\mu_i}{\frac{1}{u(z_i^{(k)})} - \sum_{\substack{j \in \mathbf{I}_\nu \\ j \neq i}} \frac{\mu_j}{z_i^{(k)} - L(z_j^{(k)})}} \quad (i \in \mathbf{I}_\nu, k = 0, 1, \dots)$$

starting from initial approximations $z_1^{(0)}, \dots, z_\nu^{(0)}$. The corresponding *single-step* method or the method in *serial mode* is given by the iterative formula:

$$(12) \quad z_i^{(k+1)} = z_i^{(k)} - \frac{\mu_i}{\frac{1}{u(z_i^{(k)})} - \sum_{j < i} \frac{\mu_j}{z_i^{(k)} - z_j^{(k+1)}} - \sum_{j > i} \frac{\mu_j}{z_i^{(k)} - L(z_j^{(k)})}} \quad (i \in \mathbf{I}_\nu, k = 0, 1, \dots).$$

3. CONVERGENCE ANALYSIS

In this section we determine the order of convergence of the total-step method (11) and the single-step method (12) using the following definition of the order of convergence introduced by ORTEGA and RHEINBOLDT [28]:

Let $\{z^{(k)}\}$ be any sequence with limit point ζ . Then the numbers

$$R_p\{z^{(k)}\} = \begin{cases} \limsup_{k \rightarrow +\infty} |z^{(k)} - \zeta|^{1/k} & \text{if } p = 1, \\ \limsup_{k \rightarrow +\infty} |z^{(k)} - \zeta|^{1/p^k} & \text{if } p > 1, \end{cases}$$

are R -factors of the sequence. If IP is an iterative process with limit point ζ , and $C(IP, \zeta)$ is the set of all sequences generated by IP which converge to ζ , then

$$R_p(IP, \zeta) = \sup \{R_p\{z^{(k)}\} \mid \{z^{(k)}\} \in C(IP, \zeta)\} \quad (1 \leq p < +\infty),$$

are called R -factors of IP at ζ . The quantity

$$O_R(IP, \zeta) = \begin{cases} \infty, & \text{if } R_p(IP, \zeta) = 0 \text{ for all } p \in [1, +\infty), \\ \inf \{p \in [1, +\infty) \mid R_p(IP, \zeta) = 1\}, & \text{otherwise} \end{cases}$$

is called the R -order of IP at ζ .

Let an iterative process IP generate n sequences $\{z_1^{(k)}\}, \dots, \{z_n^{(k)}\}$ for the approximations to the solutions ζ_1, \dots, ζ_n . In order to estimate the order of convergence of IP , one usually introduces the error-sequences

$$\varepsilon_i^{(k)} = \|z_i^{(k)} - \zeta_i\| \quad (i \in \mathbf{I}_n).$$

The convergence analysis of the iterative methods with corrections (12) needs the following assertion which is a special case of Theorem 3 given in [14]:

Theorem 1. *Given the error-recursion*

$$(13) \quad \varepsilon_i^{(k+1)} \leq \alpha_i \prod_{j=1}^n (\varepsilon_j^{(k)})^{p_{ij}}, \quad (i \in \mathbf{I}_n, k \geq 0),$$

where $p_{ij} \geq 0$, $\alpha_i > 0$, $1 \leq i, j \leq n$. Denote the matrix of exponents appearing in (13) with P_n , that is, $P_n = [p_{ij}]_{n \times n}$. If the non-negative matrix P_n has the spectral radius $\rho(P_n) > 1$ and the corresponding eigenvector $\mathbf{x}_\rho > 0$, then all sequences $\{\varepsilon_i^{(k)}\}$ ($i \in \mathbf{I}_n$) have the R -order at least $\rho(P_n)$.

We also need the following assertion.

Lemma 1. *Let $p, q \geq 1$ be positive integers and $n \geq 2$. The sequence $\{x_n\}$ of the positive roots of the equation*

$$(14) \quad x^n - xq^{n-1} - pq^{n-1} = 0$$

is monotonically decreasing.

Proof. According to Descartes' rule of sign, the equation (14) has the unique positive root. Denote this root with x_n and define the polynomial

$$\phi_n(x) = x^n - xq^{n-1} - pq^{n-1}.$$

We have

$$\phi_n(q) = -pq^{n-1} < 0, \quad \phi_n(p+q) = (p+q)^n - 2pq^{n-1} - q^n > 0,$$

which means that $x_n \in (q, p + q)$. Since $\phi_n(x_n) = 0$, we have $x_n^n = x_n q^{n-1} + p q^{n-1}$ so that

$$\begin{aligned}\phi_{n+1}(x_n) &= x_n^{n+1} - x_n q^n - p q^n = x_n(x_n q^{n-1} + p q^{n-1}) - x_n q^n - p q^n \\ &= q^{n-1}(x_n^2 - (q-p)x_n - p q) = q^{n-1}(x_n + p)(x_n - q) > 0\end{aligned}$$

since $x_n > q$. Hence, and having in mind that the functions $\phi_j(x)$ are monotonically increasing for $x > q/n^{1/(n-1)}$, we have $x_{n+1} < x_n$ (see Figure 1). \square

Now we can state the convergence theorem for the single-step method (12).

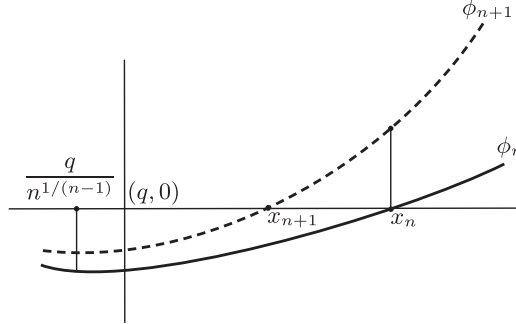


Figure 1. Geometry of polynomial roots

Theorem 2. *If initial approximations $z_1^{(0)}, \dots, z_\nu^{(0)}$ are sufficiently close to the distinct roots $\zeta_1, \dots, \zeta_\nu$ of a given polynomial, then for the R-order of convergence of the simultaneous method (12) we have*

$$r_\nu(12) := O_R((12)) \geq 2 + \tau_\nu > 6,$$

where τ_ν is the unique positive root of the polynomial $g_\nu(\tau) = \tau^\nu - 4^{\nu-1}\tau - 2^{2\nu-1}$. Furthermore, the sequence $\{r_\nu(12)\}$ is monotonically decreasing for $\nu \geq 2$, and $r_\nu \rightarrow 6$ when $\nu \rightarrow +\infty$.

Proof. Let

$$d^{(k)} = \min_{\substack{i, j \in \mathbf{I}_\nu \\ i \neq j}} \{|z_i^{(k)} - \zeta_j|, |z_i^{(k)} - z_j^{(k+1)}|, |z_i^{(k)} - L(z_j^{(k)})|\}.$$

Under the conditions of Theorem 2 the inequality

$$(15) \quad |\varepsilon^{(k)}| = \max_{1 \leq i \leq \nu} |\varepsilon_i^{(k)}| < \frac{d^{(k)}}{\sqrt{n-1}}$$

holds for each $k = 0, 1, \dots$, since $|\varepsilon^{(k)}|$ is sufficiently small quantity which tends to 0, while $d^{(k)}$ tends to the constant non-zero value $\min_{i \neq j} |\zeta_i - \zeta_j|$. The condition (15) will be utilized later in the convergence analysis.

For simplicity, we often omit the iteration index k in the iterative formulae (11) and (12). Instead of $L(z_j)$ we often write z_j^* . Let us introduce the errors of approximations

$$\varepsilon_j = z_j - \zeta_j, \hat{\varepsilon}_j = \hat{z}_j - \zeta_j, \varepsilon_j^* = L(z_j) - \zeta_j = z_j^* - \zeta_j.$$

According to the conditions of Theorem 2, we can assume that $\varepsilon_i = \mathcal{O}_M(\varepsilon_j)$ for any pair $i, j \in \mathbf{I}_\nu$. Let $\varepsilon \in \{\varepsilon_1, \dots, \varepsilon_n\}$ be the error of maximal modulus with $\varepsilon_j = \mathcal{O}_M(\varepsilon)$ ($j \in \mathbf{I}_\nu$).

Introduce the abbreviations

$$\hat{S}_i = \sum_{j < i} \frac{\mu_j \hat{\varepsilon}_j}{(z_i - \zeta_j)(z_i - \hat{z}_j)}, \quad S_i^* = \sum_{j > i} \frac{\mu_j \varepsilon_j^*}{(z_i - \zeta_j)(z_i - z_j^*)}, \quad (i \in \mathbf{I}_\nu).$$

Starting from (12) and taking into account the identity (2), we find

$$\begin{aligned} \hat{z}_i &= z_i - \frac{\mu_i}{\frac{\mu_i}{\varepsilon_i} + \left(\sum_{j < i} \frac{\mu_j}{z_i - \zeta_j} - \sum_{j < i} \frac{\mu_j}{z_i - \hat{z}_j} \right) + \left(\sum_{j > i} \frac{\mu_j}{z_i - \zeta_j} - \sum_{j > i} \frac{\mu_j}{z_i - z_j^*} \right)} \\ &= z_i - \frac{\mu_i}{\frac{\mu_i}{\varepsilon_i} - \hat{S}_i - S_i^*}. \end{aligned}$$

According to the last relation we have

$$(16) \quad \hat{\varepsilon}_i = \hat{z}_i - \zeta_i = \varepsilon_i - \frac{\mu_i \varepsilon_i}{\mu_i - \varepsilon_i \hat{S}_i - \varepsilon_i S_i^*} = -\frac{\varepsilon_i^2 (\hat{S}_i + S_i^*)}{\mu_i - \varepsilon_i \hat{S}_i - \varepsilon_i S_i^*}, \quad (i \in \mathbf{I}_\nu).$$

Since \hat{z}_j and z_j^* are more accurate approximations than z_i , it is obvious $|\varepsilon| \geq |\varepsilon_i| > |\varepsilon_i^*| > |\hat{\varepsilon}_i|$. Assuming that the errors ε_i are sufficiently small in magnitude (the assumption of Theorem 2) and having in mind the last ε -inequalities and the definition of d , we obtain

$$(17) \quad |\hat{S}_i| \leq \sum_{j < i} \frac{\mu_j |\hat{\varepsilon}_j|}{|z_i - \zeta_j| |z_i - \hat{z}_j|} \leq \frac{1}{d^2} \sum_{j < i} \mu_j |\hat{\varepsilon}_j| \leq \frac{|\varepsilon|}{d^2} \sum_{j < i} \mu_j,$$

$$(18) \quad |S_i^*| \leq \sum_{j > i} \frac{\mu_j |\varepsilon_j^*|}{|z_i - \zeta_j| |z_i - z_j^*|} \leq \frac{1}{d^2} \sum_{j > i} \mu_j |\varepsilon_j^*| \leq \frac{|\varepsilon|}{d^2} \sum_{j > i} \mu_j.$$

Hence

$$(19) \quad \begin{aligned} \frac{1}{|\mu_i - \varepsilon_i \hat{S}_i - \varepsilon_i S_i^*|} &\leq \frac{1}{\mu_i - |\varepsilon_i \hat{S}_i| - |\varepsilon_i S_i^*|} \leq \frac{1}{\mu_i - \frac{|\varepsilon|^2}{d^2} (n - \mu_i)} \\ &\leq \frac{1}{1 - \frac{|\varepsilon|^2}{d^2} (n - 1)} =: q. \end{aligned}$$

Note that $q > 0$ due to (15).

Using (17), (18) and (19), we find from (16)

$$(20) \quad |\hat{\varepsilon}_i| \leq q|\varepsilon_i|^2 \left(\sum_{j<i} |\hat{S}_i| + \sum_{j>i} |S_i^*| \right), \quad (i \in \mathbf{I}_\nu).$$

Let $\mu = \max_{1 \leq i \leq \nu} \mu_i$ and $|c_i| \leq C$ for all $i \in \mathbf{I}_\nu$ (see Remark 1 and (10)). For two complex quantities x and y such that $x = \mathcal{O}_M(y)$ we will write $x \sim y$, emphasizing that x and y have magnitude of the same order. Then, neglecting higher-order terms, we obtain from (20)

$$(21) \quad |\hat{\varepsilon}_i| \sim \frac{\mu q}{d^2} |\varepsilon_i|^2 \left(\sum_{j<i} |\hat{\varepsilon}_j| + C \sum_{j>i} |\varepsilon_j|^4 \right), \quad (i \in \mathbf{I}_\nu).$$

Replace

$$|\varepsilon_i| = C^{-1/3} h_i, \quad |\hat{\varepsilon}_i| = C^{-1/3} \hat{h}_i, \quad |\varepsilon_i^*| = C^{-1/3} h_i^* \quad (i \in \mathbf{I}_\nu)$$

into (21) and put $h = \max_{1 \leq i \leq \nu} h_i$, $h = \mathcal{O}(\hat{h}_i)$. Note that positive entries h_i , \hat{h}_i and h_i^* form null-sequences since initial approximations are assumed to be close enough to the roots. Now (21) becomes

$$\hat{h}_i \sim \frac{\mu q C^{-2/3}}{d^2} h_i^2 \left(\sum_{j<i} \hat{h}_j + \sum_{j>i} h_j^4 \right), \quad (i \in \mathbf{I}_\nu),$$

or in the form

$$(22) \quad \hat{h}_i \sim h_i^2 \left(\sum_{j<i} \hat{h}_j + \sum_{j>i} h_j^4 \right), \quad (i \in \mathbf{I}_\nu)$$

since $\mu q C^{-2/3} / d^2$ is the bounded quantity.

Let us consider first the case $\nu = 2$. From (22) we have

$$\hat{h}_1 \sim h_1^2 h_2^4, \quad \hat{h}_2 \sim h_1^2 h_2^6.$$

Introducing the iteration index, these relations can be written in the form

$$h_1^{(k+1)} \sim (h_1^{(k)})^2 (h_2^{(k)})^4, \quad h_2^{(k+1)} \sim (h_1^{(k)})^2 (h_2^{(k)})^6.$$

According to Theorem 1, for these relations we form the matrix of exponents

$$P_2 = \begin{bmatrix} 2 & 4 \\ 2 & 6 \end{bmatrix}$$

with the spectral radius $\rho(P_2) = 4 + 2\sqrt{3} \approx 7.464$ and the corresponding eigenvector $\mathbf{x}_\rho = (\sqrt{3} - 1, 1) > 0$. Hence, in regard to Theorem 1, we obtain

$$O_R((12))_{\nu=2} \geq \rho(P_2) = 4 + 2\sqrt{3} \approx 7.464.$$

the analysis given in [2] it can be shown that the lower bound of the R -order of the single-step method (12), denoted by $O_R((12), \zeta)$, for which the relations (22) hold, is given by the spectral radius $\rho(\mathcal{A}_\nu)$. The spectral radius $\rho(\mathcal{A}_\nu)$ is the unique positive root of the characteristic polynomial of the matrix \mathcal{A}_ν given by

$$\phi_\nu(\lambda) = (\lambda - 2)^\nu - 4^{\nu-1}(\lambda - 2) - 2^{2\nu-1}.$$

Substituting $\tau = \lambda - 2$ we obtain the polynomial

$$(24) \quad \psi_\nu(\tau) = \phi_\nu(\tau + 2) = \tau^\nu - 4^{\nu-1}\tau - 2^{2\nu-1}.$$

Observe that the equation (24) is a particular case of (14) for $p = 2$ and $q = 4$. Therefore, the sequence $\{\tau_\nu\}$ of positive roots of (24) is monotonically decreasing and $\tau_\nu > 4$ in regard to Lemma 1. Since $\tau_3 \approx 4.766 < \tau_2 \approx 5.464$, it follows that the sequence $\{\tau_\nu\}$ is monotonically decreasing for $\tau_\nu \geq 2$. Hence, $O_R((12)) > 6$ and the sequence of the lower bounds $\{O_R((12))_{\nu \geq 2}\}$ is monotonically decreasing with $\nu \geq 2$.

Let r_ν be the R -order of convergence of a simultaneous method for multiple roots for which the relations

$$h_i \sim h_i^p \left(\sum_{j < i} \hat{h}_j + \sum_{j > i} h_j^q \right) \quad (p, q \geq 1, i = 1, \dots, \nu)$$

are valid. Using DEUTSCH's result on the spectral radius of irreducible matrices [6], it was proved in [30, p. 9] that

$$r_\nu > p + q + \frac{pq}{(\nu - 1)(p + q)}.$$

Hence $r_\infty = \lim_{\nu \rightarrow \infty} r_\nu = p + q$. In our specific case we have $p = 2$, $q = 4$ and so that $r_\infty = 6$. Therefore, $O_R((12)) \in (6, 8)$ for $\nu \geq 2$ and tends to 6 when $\nu \rightarrow \infty$. \square

The values of the lower bound of the R -order of the single-step method (12) are given in Table 1 for several values of the number of distinct roots ν .

ν	2	3	4	5	6	7	10	15	20
R -order	7.464	6.766	6.520	6.393	6.316	6.265	6.178	6.115	6.085

Table 1. The lower bounds of R -order of convergence of the single-step method (12).

In the case of the total-step method (11), the first sum in (21) does not exist so that we have (taking iteration index)

$$(25) \quad \varepsilon_i^{(k+1)} = A_i^{(k)} (\varepsilon^{(k)})^6 + \mathcal{O}_M((\varepsilon^{(k)})^7),$$

where $A_j^{(m)} \neq 0$ is a constant of the bounded magnitude for all $m \geq 0$. According to the results given by ORTEGA and RHEINBOLDT in [28, pp. 297–298], it can be proved that the R -order of the total-step method (11) is *at least six*. To be more

precise regarding the convergence rate of the iterative method (11), we use Exercise 9.3-4 in [28, p. 298] given here in the form of

Proposition 1. *Suppose that for the sequence $\{\mathbf{x}^{(k)}\} \subset \mathbf{R}^n$ with limit \mathbf{x}^* there is some $s \in [1, \infty)$ such that the limit*

$$C_s\{\mathbf{x}^{(k)}\} = \lim_{k \rightarrow +\infty} \frac{\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\|}{\|\mathbf{x}^{(k)} - \mathbf{x}^*\|^s}$$

exists and $0 < C_s < \infty$. Then the R -order of $\{\mathbf{x}^{(k)}\}$ is exactly s .

Now we can state the following assertion.

Theorem 3. *The R -order of the total-step method (11) is exactly six.*

Proof. We deal with ν ($\leq n$) distinct roots. For simplicity, we take the norm

$$\|\mathbf{z}\|_1 = \sum_{j=1}^{\nu} |z_j|, \quad \mathbf{z} = (z_1, \dots, z_{\nu}) \text{ and obtain by (25)}$$

$$(26) \quad \frac{\|\mathbf{z}^{(k+1)} - \boldsymbol{\zeta}\|}{\|\mathbf{z}^{(k)} - \boldsymbol{\zeta}\|^6} = \frac{\sum_{j=1}^{\nu} |A_j^{(k)} (\varepsilon^{(k)})^6 + O_M((\varepsilon^{(k)})^7)|}{\left(\sum_{j=1}^{\nu} |\varepsilon_j^{(k)}|\right)^6}.$$

Let $\underline{A}^{(k)} = \min_{1 \leq j \leq \nu} |A_j^{(k)}|$ with the limit $\lim_{k \rightarrow \infty} \underline{A}^{(k)} = A$, $0 < A < \infty$. Then, after dividing by $|\varepsilon_1^{(k)}|^6$, (26) becomes

$$(27) \quad \frac{\|\mathbf{z}^{(k+1)} - \boldsymbol{\zeta}\|}{\|\mathbf{z}^{(k)} - \boldsymbol{\zeta}\|^6} \geq \underline{A}^{(k)} \cdot \frac{|1 + O_M(\varepsilon^{(k)})| \left(1 + (\nu - 1) \left|\frac{\varepsilon^{(k)}}{\varepsilon_1^{(k)}}\right|^6\right)}{\left(1 + \sum_{j=2}^{\nu} \left|\frac{\varepsilon_j^{(k)}}{\varepsilon_1^{(k)}}\right|\right)^6}.$$

Assume that initial approximations $z_1^{(0)}, \dots, z_{\nu}^{(0)}$ to the roots are chosen so that the errors $|z_j^{(0)} - \zeta_j|$ are of approximately the same magnitude. Then all quotients $|\varepsilon^{(k)}/\varepsilon_1^{(k)}|$, $|\varepsilon_j^{(k)}/\varepsilon_1^{(k)}|$ will be bounded at each iteration with limits, say,

$$+\infty > \lim_{k \rightarrow \infty} |\varepsilon^{(k)}/\varepsilon_1^{(k)}| = \eta_1 > 0 \quad \text{and} \quad +\infty > \lim_{k \rightarrow \infty} |\varepsilon_j^{(k)}/\varepsilon_1^{(k)}| = \eta_{j1} > 0.$$

Note that $\eta_1 = \max_{1 \leq j \leq \nu} \eta_{j1}$. Taking the limit in (27) we find

$$C_3((11)) = \lim_{k \rightarrow +\infty} \frac{\|\mathbf{z}^{(k+1)} - \boldsymbol{\zeta}\|}{\|\mathbf{z}^{(k)} - \boldsymbol{\zeta}\|^6} \geq A \cdot \frac{1 + (\nu - 1)\eta_1^6}{\left(1 + \sum_{j=2}^{\nu} \eta_{j1}\right)^6} > 0.$$

Therefore, the asymptotic error constant $C_3((11))$ is strictly positive. Hence, according to Proposition 1 it follows that the R -order of the iterative method (11) is exactly six.

4. COMPUTATIONAL ASPECTS

From a practical point of view, it is of great importance to estimate the computational efficiency of any iterative root-finding method since it is closely connected to the features such as the number of necessary numerical operations in computing the roots with the required accuracy, the convergence speed, processor time of a computer, etc. The knowledge of the computational efficiency is of particular interest in designing a package of root-solvers. More details about this topic may be found in [30, Ch. 6].

In this section we compare the convergence behavior and computational efficiency of the single-step methods (7), (8) and the new simultaneous method (12). This comparison procedure is entirely justified since the analysis of efficiency given in [30, Ch. 6] for several computing machines showed that the method (8) has the highest computational efficiency in the class of simultaneous methods based on fixed point relations. Note that we did not compare the new method (12) to other sixth-order methods for the two reasons:

1) Very efficient methods of the form (4) do not exist (except the methods (11) and (12)); in fact, LI-LIAO-CHENG's two-point method (9), used in the construction of (11) and (12), was the first optimal two-point method for multiple roots, launched in 2009. Although some generalized optimal two-point methods appeared later, LI-LIAO-CHENG's method has the simplest form.

2) The existing sixth-order methods for multiple roots, such as Halley-like methods [31], [42], are less efficient than the analyzed method (8), see [30, Ch. 6].

Comparing the iterative formulas (8) and (12) we observe that the new formula (12) requires ν new polynomial evaluations per iterations in relation to (8). Hence we conclude that the minimal computational efficiency of the iterative method (12) appears when $\nu = n$, that is, when all roots are simple. For this reason we will consider this "worst case" in our computational analysis.

As presented in [22, Ch. 1] and [30, Ch. 6], the efficiency of an iterative method (IM) can be successfully estimated using the *efficiency index* given by

$$(28) \quad E(\text{IM}) = \frac{\log r}{\theta},$$

where r is the R -order of convergence of the iterative method IM, and θ is its computational cost. The rank list of methods obtained by this formula mainly matches well with a real CPU (central processor unit) time.

In order to evaluate the computation cost θ it is preferable to use the number of arithmetic operations per iteration taken with certain *weights* depending on the execution times of operations. Denote these weights with w_{as} , w_m and w_d for addition/subtraction, multiplication, and division, respectively. Let $AS(n)$, $M(n)$ and $D(n)$ be the number of additions+subtractions, multiplications and divisions per one iteration for all n roots of a given polynomial of degree n . Then the computational cost θ can be (approximately) expressed as

$$(29) \quad \theta = \theta(n) = w_{as}AS(n) + w_mM(n) + w_dD(n)$$

and from (28) and (29) we obtain

$$(30) \quad E(\text{IM}, n) = \frac{\log r}{w_{as}AS(n) + w_mM(n) + w_dD(n)}.$$

For more details see [29].

We consider real polynomials with real roots for simplicity. The analysis for complex polynomials is very similar although it is slightly tedious since either a reduction to operations in real arithmetic is required, or the weights for complex operations are necessary (using, for instance, recent results given in [4]). The numbers of basic operations in real arithmetic are given in Table 2 as functions of the polynomial degree n .

Methods	$AS(n)$	$M(n)$	$D(n)$
Ehrlich-Aberth's type method (7)	$4n^2 - 2n$	$2n^2$	$n^2 + n$
Nourein's type method (8)	$4n^2 - n$	$2n^2$	$n^2 + n$
The new method (12)	$5n^2 + n$	$3n^2 + 2n$	$n^2 + 2n$

Table 2. The number of basic operations (real arithmetic operations)

To compare the simultaneous methods (7), (8) and (12), we used the weights (appearing in (30)) determined according to the estimation of complexity of basic operations in multi-precision arithmetic. Without loss of generality, we assume that floating-point number representation is used, with a binary fraction of b bits. In other words, we deal with “precision b ” binary digits, giving results with a relative error of approximately 2^{-b} . Following results given in [4], the execution time $t_b(A)$ and $t_b(S)$ of addition and subtraction is $\mathcal{O}(b)$. Using SCHÖNHAGE-STRASSEN multiplication (see [9], [38]), often implemented in multi-precision libraries (used, for instance, in the computer algebra systems *Mathematica*, *Maple*, *Magma*), we have $t_b(M) = \mathcal{O}(b \log b \log \log b)$ and $t_b(D) = 3.5t_b(M)$. We chose the weights w_{as} , w_m and w_d proportional to $t_b(A)$, $t_b(M)$ and $t_b(D)$, respectively, for a 128-bit architecture ($b = 128$).

Applying (30) we calculated the percent ratios

$$\rho_{11,6}(n) = (E((11), n)/E((6, n) - 1) \cdot 100 \text{ (in \%)}), \quad (\text{New/EA}\%)$$

$$\rho_{11,7}(n) = (E((11), n)/E((7), n) - 1) \cdot 100 \text{ (in \%)}), \quad (\text{New/N}\%)$$

where EA, N and New stand for the method (7) of Ehrlich-Aberth's type, the method (8) of Nourein's type and the new method (12) with Li-Liao-Cheng's corrections, respectively. These ratios are graphically displayed in Figure 2 as the functions of the polynomial degree n and show the (percentage) improvement of computational efficiency of the new method (12) in relation to the methods (7) and (8). In Figure 2 $\rho_{11,6}(n)$ is drawn by dotted line and $\rho_{11,7}(n)$ by full line. Note that very similar curves are obtained using the weights proportional to the execution times of basic operations for octuple precision (machine epsilon $\sim 10^{-67}$) for Pentium M 2.8 GHz running Fedora core 3 and Opteron 64-bit processor (data taken from [10]).

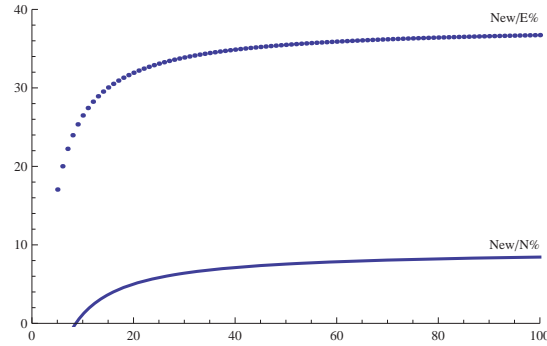


Figure 2. Ratios of efficiency indices

It is evident from Figure 2 that the new method (12) is more efficient than the methods (7) and (12), especially for large n . For polynomials of low degree the methods (12) and (8) are competitive. However, dealing with multiple roots, when $\nu < n$, the mentioned dominance of the method (12) is significant since the computational cost of the methods (11) and (12) is decreased. The improvement is especially expressive in regard to the method (7) of Ehrlich-Aberth's type (New/EA% - dotted line). Having in mind the mentioned fact on the dominant efficiency of the Nourain-like method, it follows that the proposed family of simultaneous methods (12) is the *most efficient method* for the simultaneous determination of polynomial multiple roots in the class of methods based on fixed point relations.

To demonstrate the convergence behavior of the total-step methods (5), (6), (11) and the single-step methods (7), (8) and (12), we have tested a number of polynomial equations; for illustration, among a number of tested algebraic polynomials we have selected three examples. To present the results of the third iteration, we have applied the programming package *Mathematica* with multi-precision arithmetic relying on the GNU multiple-precision package GMP developed by Granlund [12]. The comparison of the newly proposed methods (11) and (12) with a quadratically one-root-at-time method is presented in Example 3 in Section 4.

As a measure of accuracy of the obtained approximations, we calculated Euclid's norm

$$e^{(k)} := \|\mathbf{z}^{(k)} - \boldsymbol{\zeta}\|_2 = \left(\sum_{i=1}^{\nu} |z_i^{(k)} - \zeta_i|^2 \right)^{1/2} \quad (k = 0, 1, \dots),$$

where $\mathbf{z}^{(k)} = (z_1^{(k)}, \dots, z_\nu^{(k)})$ and $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_\nu)$. In order to find suitable initial approximations we have used inclusion disks $\{z : |z| \leq R\}$, centered at the origin, that contain all roots of a given polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$. Here the radius R is determined using one of 45 formulae (as functions of polynomial coefficients) given in the book [22, Sec.1.10]. One of the often used formulae is given by

$$R = 2 \max_{1 \leq \lambda \leq n} |a_{n-\lambda}|^{1/\lambda},$$

which can be found in many papers and books (see, e.g., [13]). Complete searching multi-stage procedures have been described, e.g., in [8] and [35].

EXAMPLE 1. The total-step methods (5), (6), (11) and their single-step variants (7), (8) and (12) were applied for the simultaneous approximation to the roots of the polynomial

$$f_{13}(z) = (z - 2)^3(z^2 + 1)^5.$$

The roots of this polynomial are 2 and $\pm i$ with the multiplicities 3 and 5, respectively. The initial approximations were selected to be

$$z_1^{(0)} = 2.3 - 0.3i, \quad z_2^{(0)} = 0.3 + 1.3i, \quad z_3^{(0)} = 0.3 - 1.3i.$$

The entries of the errors obtained in the first three iterations are given in Table 3, where the denotation $A(-q)$ means $A \times 10^{-q}$. The error of the starting points is $e^{(0)} \approx 0.735$.

Total-step methods \rightarrow	(5)	(6)	(11)
$e^{(1)}$	2.53(-2)	9.39(-3)	1.59(-3)
$e^{(2)}$	1.62(-6)	5.14(-10)	3.42(-19)
$e^{(3)}$	3.33(-18)	4.34(-39)	5.64(-113)
Single-step methods \rightarrow	(7)	(8)	(12)
$e^{(1)}$	1.52(-2)	7.88(-3)	7.81(-4)
$e^{(2)}$	3.20(-7)	1.30(-10)	2.20(-21)
$e^{(3)}$	7.67(-22)	9.82(-48)	5.50(-145)

Table 3. Euclid's norm of the errors – Example 1

EXAMPLE 2. The same methods were applied for the simultaneous approximation to the roots of the polynomial

$$\begin{aligned} f_{20}(z) = & z^{20} + 4z^{19} - 20z^{18} - 72z^{17} + 252z^{16} + 664z^{15} - 2092z^{14} - 3440z^{13} \\ & + 12450z^{12} + 9520z^{11} - 51476z^{10} - 1264z^9 + 142360z^8 - 82488z^7 \\ & - 228612z^6 + 279376z^5 + 117237z^4 - 337300z^3 + 77400z^2 + 135000z - 67500. \end{aligned}$$

The roots of this polynomial are $-1, -3, 1 \pm i, 1, \pm 2 \pm i$, with the multiplicities 2, 3, 2, 2, 3, 2, 2, 2, 2, respectively. The starting approximations were ($e^{(0)} \approx 0.85$)

$$\begin{aligned} z_1^{(0)} &= -1.2 + 0.2i, & z_2^{(0)} &= -2.8 - 0.2i, & z_3^{(0)} &= 1.2 + 1.2i, \\ z_4^{(0)} &= 1.2 - 1.2i, & z_5^{(0)} &= 0.8 - 0.2i, & z_6^{(0)} &= 1.8 + 1.2i, \\ z_7^{(0)} &= 1.8 - 1.2i, & z_8^{(0)} &= -2.2 + 0.8i, & z_9^{(0)} &= -2.2 - 0.8i. \end{aligned}$$

The entries of the maximal errors obtained in the first three iterations are given in Table 4.

Total-step methods \rightarrow	(5)	(6)	(11)
$e^{(1)}$	1.11(-1)	7.13(-2)	4.72(-2)
$e^{(2)}$	2.03(-4)	3.16(-6)	4.21(-9)
$e^{(3)}$	2.49(-12)	3.44(-23)	3.95(-51)
Single-step methods \rightarrow	(7)	(8)	(12)
$e^{(1)}$	8.48(-1)	5.59(-2)	4.15(-2)
$e^{(2)}$	6.47(-5)	7.61(-7)	2.95(-9)
$e^{(3)}$	5.10(-15)	7.23(-27)	2.75(-52)

Table 4. Euclid's norm of the errors – Example 2

EXAMPLE 3. In order to find the roots of the polynomial

$$\begin{aligned} f_{18}(z) = & z^{18} + (2 - 2i)z^{17} - 14z^{16} - (18 - 26i)z^{15} + (80 - 12i)z^{14} \\ & + (26 - 118i)z^{13} - (238 - 136i)z^{12} + (146 + 182i)z^{11} + (307 - 476i)z^{10} \\ & - (380 - 160i)z^9 + (236 + 320i)z^8 + (32 - 712i)z^7 - (804 - 880i)z^6 \\ & + (512 + 96i)z^5 - (80 + 832i)z^4 - (1024 - 1152i)z^3 - (448 - 256i)z^2 \\ & - (1024 - 512i)z + (-768 + 1024i) \end{aligned}$$

we applied the same methods as in Examples 1 and 2.

Total-step methods →	(5)	(6)	(11)
$e^{(1)}$	8.75(-2)	4.76(-2)	2.36(-2)
$e^{(2)}$	2.44(-4)	1.13(-6)	1.17(-10)
$e^{(3)}$	2.02(-12)	7.33(-25)	1.05(-61)
Single-step methods →	(7)	(8)	(12)
$e^{(1)}$	5.33(-2)	3.45(-2)	2.19(-2)
$e^{(2)}$	4.61(-5)	4.62(-7)	1.25(-10)
$e^{(3)}$	1.20(-14)	3.33(-28)	2.60(-67)
Schröder's method →	(31)		
$e^{(1)}$	0.463		
$e^{(2)}$	0.292		
$e^{(3)}$	0.203		
$e^{(4)}$	0.236		
$e^{(5)}$	0.152		
$e^{(6)}$	0.141		
$e^{(7)}$	5.05(-2)		
$e^{(8)}$	8.64(-3)		
$e^{(9)}$	2.54(-4)		
$e^{(10)}$	2.13(-7)		
$e^{(11)}$	1.50(-13)		
$e^{(12)}$	7.43(-26)		

Table 5. Euclid's norm of the errors – Example 3

The roots of the polynomial $f_{18}(z)$ are $-1, -2, 2, 1 \pm i, \pm i, -2 + i$ with the respective multiplicities 2, 3, 3, 2, 2, 2, 2. The following starting approximations were selected ($e^{(0)} = 0.8$)

$$\begin{aligned} z_1^{(0)} &= -1.2 + 0.2i, & z_2^{(0)} &= -2.2 + 0.2i, & z_3^{(0)} &= 2.2 - 0.2i, & z_4^{(0)} &= 1.2 + 1.2i, \\ z_5^{(0)} &= 1.2 - 1.2i, & z_6^{(0)} &= -0.2 + 0.8i, & z_7^{(0)} &= 0.2 - 0.8i, & z_8^{(0)} &= -2.2 + 0.8i. \end{aligned}$$

For comparison purpose, we have also applied Newton's like (or Schröder's) method for multiple roots

$$(31) \quad z_i^{(k+1)} = z_i^{(k)} - \mu_i \frac{f(z_i^{(k)})}{f'(z_i^{(k)})} \quad (k = 0, 1, \dots, i \in \mathbf{I}_\nu),$$

The stopping criterions have been given by the inequalities $e^{(k)} < \tau_1 = 10^{-9}$ and $e^{(k)} < \tau_2 = 10^{-24}$. The entries of the maximal errors obtained in the first three iterations are given in Table 5.

From Table 5 we observe that the new methods (11) and (12) fulfilled the first criterion ($\tau_1 = 10^{-9}$) after the second iteration. Since the order of (11) is six, it could be expected that quadratically convergent Schröder's method (31) will satisfy this criterion after 6 iterations (exactly, three times more). However, it required even 11 iterations. The reason for this poor result is too slow convergence of particular Schröder's sequences, which increases the total errors $e^{(k)}$ in the first iterations. Faster (in fact, quadratical) convergence of the method (31) begins after 7th iteration.

To fulfil the second termination criterion $\tau_2 = 10^{-24}$ the proposed methods (11) and (12) require 3 iterations, while Schröder's method meets this criterion after 12 iterations instead of (theoretically) expected 9. Even slower methods (6) (of order four) and (7) have satisfied the second criterion after the third iteration. Theoretically expected iterations of Schröder's method in this case had to be 6.

Both experiments point to the better convergence properties of simultaneous methods compared to one-root-at-time Schröder's method, as discussed in the Introduction. Besides this preference, we recall the advantage of simultaneous methods of lending themselves to parallel computation.

From Tables 3–5 and a number of tested polynomial equations we can conclude that the proposed methods (11) and (12) produce approximations of considerably high accuracy; two iterative steps are usually sufficient in solving most practical problems when initial approximations are reasonably close to the roots. The results of the third iteration are given only to demonstrate very fast convergence of the new methods (11) and (12). The presented analysis of computational efficiency shows that the methods (11) and (12) are more efficient than all previously developed methods for multiple roots based on fixed point relations.

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Department of Mathematics,
Faculty of Electronic Engineering,
University of Niš, 18000 Niš
Serbia
E-mails: miodragpetkovic@gmail.rs
jovana.dzunic@elfak.ni.ac.rs

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Department of Mathematics,
Faculty of Mechanical Engineering,
University of Niš, 18000 Niš
Serbia
E-mail: ljiljana@masfak.ni.ac.rs