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ON THE GENERATION OF BIPARTITE GRIDS IN 2 AND 3 DIMENSIONAL REGIONS

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We present a procedure to generate bipartite grids for simply connected domains in 2-D and 3-D, of prescribed size and controlled regularity elements. The mesh elements K of the triangulation satisfy $\zeta_K \leq C$, where ζ_K is the regularity and C is a constant depending on the shape parameters of the initial mesh. Bipartite grids permit a well-posed mixed-mixed variational formulation of problems, such as the porous media flow equation and other linear physical phenomena, as well as Galerkin-type discontinuous relaxations.

1. INTRODUCTION

The generation of quality shaped grids for geometric domains is a vast and active research field of mathematics. Defining a grid is one of the key steps in solving partial differential equations with finite element methods. However, the approximation estimates depend not only on the size of a mesh, but also on the quality shape of its elements. In addition, other types of numerical difficulties can be introduced in the method due to the quality shape of the elements. It is rather common to use triangular and rectangular elements for 2-D domains, as well as tetrahedral and hexahedral elements for 3-D domains; the choice is done according to the structure of the modeled problem. Some other questions in the field are the automatic generation of grids [**8**], the generation of polynomial patches for the approximation of two dimensional manifolds [**16**], the generation of structured and/or unstructured grids, the numerical costs and efficiency for grid generation versus the cost of computing numerical solutions of the physical problem posed on

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the grid, and the Delaunay tessellation approach for grid generation [**9, 7**]. On the other hand, the meshfree methods present a totally different approach and try to circumvent the intrinsic difficulties of mesh generation, see [**4**], although other difficulties always arise, see [**7**]. However, none of these achievements aim to fulfill the needs of a particular variational formulation.

In this work, we address the problem of generating bipartite grids. Its main motivation is to permit the mixed-mixed variational formulation (presented in [**14**]) of the porous media, partial differential equation. The mixed-mixed formulation is remarkable when modeling problems with interface transmission conditions, due to the degrees of freedom introduced in the underlying spaces of functions. Unlike any other formulation for interface problems, there are no linear coupling constraints between function spaces. This feature makes it very attractive for setting discontinuous Galerkin schemes, and analyzing problems with micro structure of fractal type. Although the formulation has been developed only for the problem of saturated flow in porous media, it can be easily extended to problems of linear elasticity, heat diffusion, etc. In the problem presented in [**14**] the domain of analysis Ω is subdivided in two, namely Ω^1 and Ω^2. The first uses the pair $[\mathbf{H}_{\text{div}}, L^2]$, and the second uses the pair $[\mathbf{L}^2, H^1]$ for the velocity and pressure, respectively. However, in that particular case it is clear how to subdivide the domain and set the pairs of modeling spaces of functions. If the formulation is to be used with another subdivision of the domain, namely the one provided by a mesh on Ω, we need to assure that each element with pairing $[\mathbf{H}_{\text{div}}, L^2]$ shares a boundary of non-negligible measure, only with elements whose pairing is of the type $[\mathbf{L}^2, H^1]$. Therefore, the graph of the mesh has to be bipartite, where two elements of the grid are connected if they have a common interface of non-negligible measure. However, it is also necessary to generate bipartite grids of prescribed size and controlled regularity, because of two main aspects. On one hand, the geometric scale at which the permeability of a porous medium may present considerable oscillations, can be very small. Therefore, physical effects of significant importance may be taking place at a small scale, which need to be addressed. On the other hand, the convergence analysis of a discretized scheme, requires to have control on the size and regularity of the elements of the grid.

Due to the limitations of the technique, we restrict our attention to simply connected domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. For simplicity, it will be also assumed that the domains are polygonal or polyhedral. This is not conceptually far from a more realistic case, because it is straightforward to generate a mesh of curved triangles (or tetrahedra) for a simply connected domain, using the gridding of a polygon (or polyhedron) inscribed in the original domain. Under the hypothesis above, the necessary and sufficient condition for triangular and tetrahedral grids to be bipartite will be presented. Additionally, a method for generating grids of arbitrary small size, with elements of controlled regularity will also be introduced. Both, the characterization of bipartite grids and the method to generate fine and regular meshes, are quite simple and easy to implement. However, it will be clear that the generation method is not optimal. In this work our main goal is to provide the theoretical setting to assure that the mixed-mixed formulation of the problem is well-posed, as well as the convergence of approximate solutions.

Next, we introduce the notation. In the following Ω and $\mathcal{O}$ indicate polygonal or polyhedral bounded open regions of $\mathbb{R}^2$ or $\mathbb{R}^3$ depending on the context; however we save Ω for the domain of analysis which is additionally simply connected. The symbol $\mathcal{O}$ will be reserved for definitions and intermediate results. Vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are denoted by bold letters. We write $\#A$ for the cardinality of the set A and $|A|$ for the length of a segment; $|A|_i$ with $i = 2, 3$ stands for the area and volume of the set, depending on the context. Triangles and tetrahedra, will typically be denoted by the letters K, L, M and Δ; since the notation is consistent they shall be seen as elements of the grid or as nodes of the grid associated graph, depending on the context. We write $\mathcal{T}$ for triangulations or tetrahedral grids of the domain and the characters $\mathcal{B}, \mathcal{Q}, \mathcal{S}$ for refinement processes of the mesh.

2. PRELIMINARIES FROM GRAPH THEORY AND TOPOLOGY

We start this section recalling the minimum background necessary from graph theory [**6**].

Definition 2.1. *Let $G = (V, E)$ be a graph*

(i) *A **walk** in G from node v_0 to node v_j is an alternating sequence*

$$W \overset{\textbf{def}}{=} \langle v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n \rangle,$$

of nodes and edges such that the endpoints of the edge e_i are v_{i-1} and v_i for all $i = 1, 2, \ldots, n$.

(ii) *A **path** is a walk with no repeated edges and no repeated nodes, except possibly the initial and final nodes.*

(iii) *A walk or path is **trivial** if it has only one node and no edges.*

(iv) *A **cycle** is a non-trivial closed path i.e., it starts and ends on the same node.*

(v) *The **length** of a walk, path or cycle is the number of edge-steps in the sequence. We denote it by $\#\langle v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n \rangle$.*

Definition 2.2. *A **self-loop** is an edge that joins a single node with itself. A **multi-edge** is a collection of two or more edges joining identical nodes. A **simple graph** has neither self-loops nor multi-edges.*

Definition 2.3. *A **cycle graph** is a single node with a self-loop or a **simple graph** whose number of nodes equals its number of edges and can be drawn so that all its nodes and edges lie on a single circle. A cycle graph having j nodes will be denoted by $C_j = \langle v_1, v_2, \ldots, v_j, v_1 \rangle$.*

Definition 2.4. *A **bipartite graph** G is a graph whose node set V can be partitioned in two subsets U, W such that each edge of G has one endpoint in U and one endpoint in W. The pair U, W is called a (node) bipartition of G, and U and W are called bipartition subsets.*

Definition 2.5. *A graph is* ***connected*** *if for every pair of nodes u and v there is a walk from u to v.*

Definition 2.6. *The* ***degree*** (*or* ***valence***) *of a node v in a graph G, denoted* $\deg(v)$, *is twice the number of self loops plus the number of proper edges joining v with any other node of G.*

Next we recall a well-known characterization result for bipartite graphs [**6**].

Theorem 2.7. *A graph G is bipartite if and only if it has no cycles of odd length.*

We close this section recalling topological basic standard results and definitions for simply connected spaces [**15**].

Definition 2.8. *Let X be a topological space.*

(i) *Given two points x, y of the space X, a* ***path*** *in X from x to y is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

(ii) *The space is said to be* ***path connected*** *if every pair of points of X can be joined by a path in X.*

(iii) *Two paths $\gamma, \gamma\prime$ mapping the interval $I = [0,1]$ into X are said to be* ***path homotopic*** *if they have the same initial point x_0 and the same final point x_1, and if there is a continuous map $F : I \times I \to X$ such that*

$$F(s,0) = \gamma(s) \quad \text{and} \quad F(s,1) = \gamma\prime(s)\,, \qquad \forall\, s \in I \tag{1a}$$

$$F(0,t) = x_0 \quad \text{and} \quad F(1,t) = x_1\,, \qquad \forall\, t \in I \tag{1b}$$

We say γ and $\gamma\prime$ are ***homotopy related*** *or simply* ***homotopic*** *and denote the relation by $\gamma \sim \gamma\prime$. Finally, we call F a* ***path homotopy*** *between γ and $\gamma\prime$.*

Lemma 2.9. *The relation $\sim$ is an equivalence relation.*

Proof. See [**15**].

Definition 2.10. *Let X be a topological space, if γ is a path in X from x_0 to x_1 and if $\gamma\prime$ is a path in X from x_1 to x_2, we define the* ***product*** *$\gamma * \gamma\prime$ of γ and $\gamma\prime$ to be the path given by*

$$\gamma * \gamma\prime\,(s) \overset{\mathbf{def}}{=} \begin{cases} \gamma(2s) & s \in \left[0, \dfrac{1}{2}\right], \\ \gamma\prime(2s-1) & s \in \left[0, \dfrac{1}{2}\right]. \end{cases}$$

Definition 2.11. *Let X be a topological space.*
(i) *Given a point $x_0 \in X$ a path in X that begins and ends at x_0 is called a* ***loop*** *based at x_0.*

(ii) *The set of path homotopy classes of loops based at x_0 with the operation $*$, is called the* ***fundamental group*** *of X relative to the* ***base point*** *x_0. It is denoted* *by* $\pi_1(X, x_0)$.

(iii) *The space is said to be* ***simply connected*** *if it is a path-connected space and* $\pi_1(X, x_0)$ *is the trivial (one-element) group for some* $x_0 \in X$, *and hence for every* $x \in X$.

3. TWO DIMENSIONAL CASE

We start this section by defining the graph associated to the triangulation of a given polygonal domain in $\mathbb{R}^2$.

Definition 3.12. *Let* $\mathcal{O} \subseteq \mathbb{R}^2$ *be an open bounded polygonal domain and* $\{K : K \in \mathcal{T}\}$ *be a triangulation of* $\mathcal{O}$. *We denote by* $\mathcal{P}$ *the set of vertices of the triangulation.*

(i) *The associated graph* $\mathcal{G}_\mathcal{T}(\mathcal{T}, \mathcal{E}_\mathcal{T})$ *is defined in the following way. The set of nodes* $\mathcal{T}$ *is defined by the set of triangles and there is an edge in* $\mathcal{E}_\mathcal{T}$ *between two elements* $K, L \in \mathcal{T}$ *if* $K \neq L$ *and* $|\partial K \cap \partial L \cap \mathcal{O}| > 0$ *i.e., if they are different elements and share a common side contained in the domain* $\mathcal{O}$, *except possibly for the extremes of the segment* $\partial K \cap \partial L$.

(ii) *An element of the triangulation* $\{K : K \in \mathcal{T}\}$ *is said to be isolated if its degree as node of* $\mathcal{G}_\mathcal{T}$ *is zero. We denote by* $\mathcal{T}_{\text{nct}}$ *the connected or non-isolated, elements of the triangulation.*

(iii) *We say a triangle is INTERIOR if* $|\partial K \cap \partial\mathcal{O}| = 0$, *and denote by* $\mathcal{T}_{\text{int}}$ *the set of interior triangles. We say a triangle* $K \in \mathcal{T}$ *is EXTERIOR if* $|\partial K \cap \partial\mathcal{O}| > 0$, *and denote by* $\mathcal{T}_{\text{ext}}$ *the set of exterior triangles.*

(iv) *We say a vertex* $\xi \in \mathcal{P}$ *is EXTERIOR if it lies on the boundary of the domain* $\partial\mathcal{O}$, *and denote the set of exterior vertices by* $\mathcal{P}_{\text{ext}}$. *We say a vertex* $\xi \in \mathcal{P}$ *of a triangle is INTERIOR if it belongs to the interior of the domain, and denote the set of interior vertices by* $\mathcal{P}_{\text{int}}$.

Remark 3.1. Notice that if a polygonal domain Ω is simply connected its graph $\mathcal{G}_\mathcal{T}$ has to be connected i.e., $\mathcal{T} = \mathcal{T}_{\text{nct}}$.

Definition 3.13. *Let* $\mathcal{O} \subseteq \mathbb{R}^2$ *be an open bounded polygonal domain and* $\{K : K \in \mathcal{T}\}$ *be a triangulation of* $\mathcal{O}$. *We say that a triangulation* $\{K : K \in \mathcal{T}\}$ *is BIPARTITE if its associated graph* $\mathcal{G}_\mathcal{T}(\mathcal{T}, \mathcal{E}_\mathcal{T})$ *is bipartite.*

Definition 3.14. *Let* $\mathcal{O} \subseteq \mathbb{R}^2$ *be an open bounded polygonal domain and* $\{K : K \in \mathcal{T}\}$ *be a triangulation of* $\mathcal{O}$.

(i) *For each* $K \in \mathcal{T}$ *we denote by* $\mathbf{b}_K$ *its centroid or center of gravity.*

(ii) *Given a cycle* $C = \langle K_1, K_2, \ldots, K_j, K_1\rangle$ *in the graph* $\mathcal{G}_\mathcal{T}$, *we denote by* γ_C *the rectifiable path generated by the sequence of segments* $[\mathbf{b}_{K_1}, \mathbf{b}_{K_2}], \ldots [\mathbf{b}_{K_{j-1}}, \mathbf{b}_{K_j}]$, $[\mathbf{b}_{K_j}, \mathbf{b}_{K_1}]$ *i.e.,* γ_C *is contained in* $\mathcal{O}$.

(iii) *Given a rectifiable path* γ *and a point* $\mathbf{b}$ *in* $\mathbb{R}^2 - \{\gamma\}$ *we denote by* $n(\gamma, \mathbf{b})$ *the winding number* [**3**].

Remark 3.2. Let C be a cycle with three or more nodes, in the associated graph $\mathcal{G}_{\mathcal{T}}$ of a triangulation $\mathcal{T}$. Notice the following

(i) The path γ_C divides the plane in only two connected components. The application $\mathbf{b} \mapsto n(\gamma_C, \mathbf{b})$ defined on $\mathbb{R}^2 - \{\gamma_C\}$ takes only three possible values: 0 on the unbounded connected component and, on the bounded component, 1 if γ_C is counterclockwise oriented or -1 if it is clockwise oriented.

(ii) Let K be an element of the triangulation and $\{\xi_i : 1 \leq i \leq 3\}$ its three vertices. Due to the definition of the path γ_C we know that $\{\xi_i : 1 \leq i \leq 3\} \cap \{\gamma_C\} = \emptyset$, hence, the winding number $n(\gamma_C, \xi)$ is well-defined for all $\xi \in \mathcal{P}$.

In order to characterize bipartite grids, first we focus on a very particular type of triangulation.

Definition 3.15. *We say that a triangulation $\{K : K \in \mathcal{T}\}$ of an open bounded simply connected polygonal domain $\mathcal{O}$ is RADIAL, if it has only one interior vertex denoted by ξ_p such that $\{K \in \mathcal{T} : \xi_p \in \partial K\} = \mathcal{T}$. In the following we refer to this interior vertex as POLE.*

Remark 3.3. Notice that since the domain $\mathcal{O}$ is simply connected, the graph of the triangulation $\mathcal{G}_{\mathcal{T}}$, must be connected.

Proposition 3.16. *Let $\{K : K \in \mathcal{T}\}$ be a radial triangulation of the simply connected domain Ω. Then*

(i) *The degree of each vertex K in the graph of the triangulation $\mathcal{G}_{\mathcal{T}}$ is 2.*

(ii) *A radial triangulation has at least three elements.*

(iii) *If $\#\mathcal{T} = j$ then $\mathcal{G}_{\mathcal{T}} = C_j$ i.e., it is a cycle graph of j vertices.*

(iv) *The triangulation is bipartite if and only if the number of triangles is even.*

Proof. (i) Let ξ_p be the pole of the triangulation and K an element of the triangulation. Since $\xi_p \in \partial K$ this implies the other two vertices of K must be exterior. If $\deg(K) = 3$ there would exist an element $L \in \mathcal{T}$ connected to K, sharing the same two exterior vertices of K. Recalling that $\mathcal{T}$ is radial, this would imply that the third vertex of L is ξ_p and the three vertices of K and L would be the same, i.e., $K = L$ which is a contradiction, hence $\deg(K) < 3$. On the other hand, since $\mathcal{G}_{\mathcal{T}}$ is connected we know $\deg(K) \geq 1$. However, if $\deg(K) = 1$ this would imply that only one side of the triangle is in the interior of Ω and the three vertices of K would be exterior, which is a contradiction since $\mathcal{T}$ is radial. Thus, $\deg(K) = 2$ and the proof of the first part is complete.

(ii) If a triangulation has less than three elements all the vertices are exterior, therefore it can not be radial.

(iii) Since $\deg(K) = 2$ for all $K \in \mathcal{T}$ then $\#\mathcal{T} = \#\mathcal{E}_{\mathcal{T}}$. On the other hand, by construction, $\mathcal{G}_{\mathcal{T}}$ is simple, therefore it must be the cycle graph C_j.

(iv) Since $\mathcal{G}_{\mathcal{T}}$ is a cycle graph it contains a unique cycle of length $\#\mathcal{T}$, then due to Theorem 2.7 it is bipartite if and only if $\#\mathcal{T}$ is even.

Definition 3.17. *Let $\mathcal{O}$ be an open, bounded, simply connected, polygonal domain, $\{K : K \in \mathcal{T}\}$ be a triangulation of $\mathcal{O}$ and ξ an interior vertex of the triangulation. We define the associated radial subgraph C_ξ by the triangles $V_\xi \stackrel{\mathbf{def}}{=} \{K \in \mathcal{T} : \xi \in \partial K\}$, and the set of edges $\mathcal{E}_\xi$ which connect two elements of V_ξ. Clearly, C_ξ is radial and due to Proposition* 3.16 *we know that $C_\xi = C_j$, where $j \stackrel{\mathbf{def}}{=} \#V_\xi = \#\{K \in \mathcal{T} : \xi \in \partial K\}$.*

Lemma 3.18. *Let $\{K : K \in \mathcal{T}\}$ be a triangulation of Ω. Then, for any cycle C in the graph $\mathcal{G}_\mathcal{T}$ and for each $K \in C$ there exists at least one vertex $\xi \in \mathcal{P}_{\text{int}} \cap \partial K$, such that the winding number $n(\gamma_C, \xi)$ is not zero.*

Proof. Since C is a cycle, the path γ_C divides the plane into two components. Let K be an element of the cycle. Since K is a triangle at least one of its vertices lies within the bounded component of the plane, namely ξ. Then $n(\gamma_C, \xi) \neq 0$ and ξ can not be an exterior vertex because Ω is simply connected and $n(\gamma_C, \mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbb{R}^2 - \Omega$, see [**3**].

Theorem 3.19. *Let $\{K : K \in \mathcal{T}\}$ be a triangulation of Ω such that it has only one interior vertex, namely $\mathcal{P}_{\text{int}} = \{\xi_p\}$. Then, the triangulation is bipartite if and only if $\#\{K \in \mathcal{T} : \xi_p \in \partial K\}$ is even.*

Proof. First notice that the triangulation has at least three elements and, as discussed in Remark 3.1, its graph $\mathcal{G}_\mathcal{T}$ is connected. If $\{K \in \mathcal{T} : \xi_p \in \partial K\} = \mathcal{T}$ there is nothing to prove due to Proposition 3.16.

If $\mathcal{T} - \{K \in \mathcal{T} : \xi_p \in \partial K\} \neq \emptyset$ consider the radial subgraph C_{ξ_p} given by Definition 3.17, thus $C_{\xi_p} = C_j$ where $j \stackrel{\mathbf{def}}{=} \#\{K \in \mathcal{T} : \xi_p \in \partial K\}$. Let L be in $\mathcal{T} - \{K \in \mathcal{T} : \xi_p \in \partial K\}$ then, it must hold that its three vertices are exterior and due to Lemma 3.18, L can not belong to any cycle. Therefore, any cycle C in the graph $\mathcal{G}_\mathcal{T}$ must lie in the subgraph C_{ξ_p}; however, the subgraph $C_{\xi_p} = C_j$ is cyclic and its unique cycle is itself. Thus, the graph $\mathcal{G}_\mathcal{T}$ contains a unique cycle and due to Theorem 2.7 it is bipartite if and only if $j = \#\{K \in \mathcal{T} : \xi_p \in \partial K\}$ is even. □

Finally, we characterize bipartite grids in the following result.

Theorem 3.20. *A triangulation $\{K : K \in \mathcal{T}\}$ of Ω is bipartite if and only if, for every interior vertex $\xi \in \mathcal{P}_{\text{int}}$, the number of incident triangles $\#\{K \in \mathcal{T} : \xi \in \partial K\}$ is even.*

Proof. We begin proving the necessity. Suppose that there exists an interior vertex, namely $\xi_p \in \mathcal{P}_{\text{int}}$, such that $\#\{K \in \mathcal{T} : \xi_p \in \partial K\} = 2j + 1$. Then the cycle $C = \langle K_1, \ldots, K_{2j+1}, K_1\rangle$, where $\xi_p \in \partial K_i$ for all $i = 1, \ldots, 2j + 1$, belongs to the graph and it has odd length. Therefore the graph can not be bipartite due to Theorem 2.7.

In order to prove the sufficiency of the condition we proceed by induction on the number of interior vertices. The case $\#\mathcal{P}_{\text{int}} = 0$ implies that the graph $\mathcal{G}_\mathcal{T}$ has no cycles due to Lemma 3.18, consequently it is bipartite according to Theorem

2.7. If $\#\mathcal{P}_{\text{int}} = 1$ the result follows due to Theorem 3.19. Assume now that the result holds whenever the number of interior vertices is less or equal than j, and let $\{K : K \in \mathcal{T}\}$ be a triangulation of Ω such that $\#\mathcal{P}_{\text{int}} = j + 1$. Define

$$\mathcal{T}_{SC} \overset{\textbf{def}}{=} \{K \in \mathcal{T} : \#(\partial K \cap \mathcal{P}_{\text{ext}}) < 3\},$$

and

$$\Omega_{SC} \overset{\textbf{def}}{=} \bigcup\{K : K \in \mathcal{T}_{SC}\}$$

i.e., the elements of the triangulation which do not have three exterior vertices, and Ω_{SC} the subdomain of Ω for which $\mathcal{T}_{SC}$ is a natural triangulation. Clearly, the number of interior vertices in $\mathcal{T}_{SC}$ equals the number of interior vertices in $\mathcal{T}$ i.e., $j + 1$. Moreover, if ξ is an interior vertex of the triangulation then

$$\#\{K \in \mathcal{T}_{SC} : \xi \in \partial K\} = \#\{K \in \mathcal{T} : \xi \in \partial K\}.$$

Recall that if $\#(\partial K \cap \mathcal{P}_{\text{ext}}) = 3$ then, due to Lemma 3.18, K can not belong to any cycle. Therefore, every cycle in $\mathcal{T}_{SC}$ is even if and only if every cycle in $\mathcal{T}$ is even. Hence, without loss of generality it can be assumed that the triangulation satisfies $\#(\partial K \cap \mathcal{P}_{\text{ext}}) < 3$ for all $K \in \mathcal{T}$.

Let K be an exterior triangle, then one of its vertices, namely ζ, must be interior and the other two exterior since only one of its sides lies on the boundary of Ω, i.e., it has degree two. From now on, we denote by K_ζ this element of $\mathcal{T}$. Consider the domain $\Omega_\zeta \overset{\textbf{def}}{=} \Omega - \text{cl}(K_\zeta)$. Clearly the domain must be simply connected since K_ζ is simply connected and it is an exterior element; also $\zeta \notin \Omega_\zeta$. On the other hand, the family $\mathcal{T}_\zeta \overset{\textbf{def}}{=} \{K \in \mathcal{T} : K \neq K_\zeta\}$ is clearly a triangulation of the domain Ω_ζ in which ζ is not an interior vertex. Hence, $\mathcal{T}_\zeta$ is a triangulation of a simply connected domain whose interior vertices are given by the set $\mathcal{P}_{\text{int}} - \{\zeta\}$, i.e., it has only j interior vertices; additionally we have

$$\#\{K \in \mathcal{T}_\zeta : \xi \in \partial K\} = \#\{K \in \mathcal{T} : \xi \in \partial K\}, \quad \forall \xi \in \mathcal{P}_{\text{int}} - \{\zeta\}.$$

We conclude that each interior vertex of the triangulation $\mathcal{T}_\zeta$ has an even number of incident triangles, due to the induction hypothesis the graph is bipartite. Denote by U_ζ, W_ζ the node bipartition of the graph $\mathcal{G}_{\mathcal{T}_\zeta}$.

Consider now the radial subgraph C_ζ given by Definition 3.17. From the hypothesis we know that C_ζ contains an even number of triangles, then the set $\{K \in \mathcal{T}_\zeta : \zeta \in \partial K\}$ has an odd number of triangles. Since $\deg(K_\zeta) = 2$, denote by $L, M \in \mathcal{T}$ the triangles such that $|\partial K_\zeta \cap \partial L| > 0$, $|\partial K_\zeta \cap \partial M| > 0$. Clearly $K, L \in \mathcal{T}_\zeta$. We claim that these triangles belong to only one subset of the node bipartition. Let $\langle L, \ldots, M\rangle$ be the unique path from L to M within both graphs C_ζ and $\mathcal{G}_{\mathcal{T}_\zeta}$. Evidently its length is the even number $\#C_\zeta - 2$. Recalling that the graph $\mathcal{G}_{\mathcal{T}_\zeta}$ is bipartite we conclude that L and M must belong to the same element of the node partition, either U_ζ or W_ζ; without loss of generality assume $L, M \in U_\zeta$. Thus, the pair $U \overset{\textbf{def}}{=} U_\zeta$, $W \overset{\textbf{def}}{=} W_\zeta \cup \{K_\zeta\}$ is a node bipartition of the graph $\mathcal{G}_{\mathcal{T}}$ since it only has one extra element, K_ζ, whose only two edges have the other endpoint on a triangle belonging to U. This completes the proof. □

Next we present a result of existence for bipartite triangulations of simply connected polygonal regions in $\mathbb{R}^2$; it is a method for refining a given grid into a bipartite one. It does not pursue advantages from the numerical point of view, only from the analytical point of view, in order to make possible the mixed-mixed variational formulation of the porous media problem.

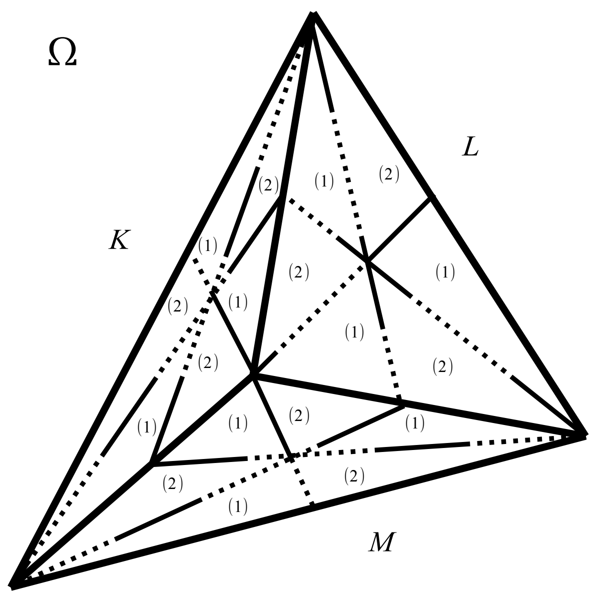

Figure 1. Bipartite Refinement

Definition 3.21. (i) *Let $K \subseteq \mathbb{R}^2$ be a triangle, we define its BIPARTITE GRIDDING as the collection of the six triangles $\{L_i : 1 \leq i \leq 6\}$ generated by the three medians of the triangle K.*

(ii) *Let $\mathcal{T} = \{K : K \in \mathcal{T}\}$ be a triangulation of the domain Ω. We define its BIPARTITE REFINEMENT, as the mesh that is generated applying the bipartite gridding process to each triangle K of the triangulation. We denote it by $\mathcal{BT} = \{L : L \in \mathcal{BT}\}$.*

Theorem 3.22. *Let $\Omega \subseteq \mathbb{R}^2$, then it has a bipartite triangulation.*

Proof. Let $\mathcal{T} = \{K : K \in \mathcal{T}\}$ be any triangulation of Ω, since the domain is polygonal such triangulation exists; let $\mathcal{BT} = \{L : L \in \mathcal{BT}\}$ be its bipartite refinement; we are to prove that this grid is bipartite. Let ξ be an interior vertex of the triangulation $\mathcal{BT}$, due to Theorem 3.20 it is enough to show that $\#\{L \in \mathcal{BT} : \xi \in \partial L\}$ is even. Notice that ξ has only three possibilities:

(i) If ξ was an interior vertex of the triangulation $\mathcal{T}$ then $\#\{K \in \mathcal{T} : \xi \in \partial K\} \neq 0$. For each triangle $K \in \mathcal{T}$ incident in ξ the median passing through ξ divides K in two sub-triangles of $\mathcal{BT}$, thus $\#\{L \in \mathcal{BT} : \xi \in \partial L\} = 2\#\{K \in \mathcal{T} : \xi \in \partial K\}$.

(ii) If $\xi = \mathbf{b}_K$ i.e., it is the centroid of some $K \in \mathcal{T}$ then $\#\{L \in \mathcal{BT} : \xi \in \partial L\} = 6$.

(iii) If ξ is the middle point of a triangle's edge K in the triangulation $\mathcal{T}$, recalling it is an interior vertex we conclude that $\#\{L \in \mathcal{BT} : \xi \in \partial L\} = 4$.

Since in the three cases the number of triangles incident in ξ is even the result follows.

Remark 3.4. In Figure 1 we illustrate the Theorem 3.22 above. Clearly, the triangulation $\mathcal{T} \overset{\mathbf{def}}{=} \{K, L, M\}$ of the domain Ω is not bipartite, however $\mathcal{BT}$ is bipartite.

We close this section presenting a method to refine bipartite grids without deteriorating the regularity of the mesh and preserving the bipartite property. First we recall the definition of the classical shape and size parameters of a grid [**5**].

Definition 3.23. *Let $\mathcal{T}$ be a triangulation of a domain Ω and $K \in \mathcal{T}$ one of its triangular elements.*

(i) *Denote by h_K its diameter.*

(ii) *Denote by $\rho_K \overset{\mathbf{def}}{=} \sup\{$diameter of $B : B$ ball contained in $K\}$.*

(iii) *Define the regularity of the triangle by the ratio $\zeta_K \overset{\mathbf{def}}{=} \dfrac{h_K}{\rho_K}$.*

(iv) *The size of a mesh h is defined by $h \overset{\mathbf{def}}{=} \max\{h_K : K \in \mathcal{T}\}$.*

(v) *The regularity of a mesh $\mathcal{T}$ is defined by $\zeta \overset{\mathbf{def}}{=} \zeta(\mathcal{T}) = \max\left\{\dfrac{h_K}{\rho_K} : K \in \mathcal{T}\right\}$.*

Next, we recall the definition of red refinement in two dimensions, see [**1**]

Definition 3.24. (i) *Let K be a triangle. We subdivide it into four geometrically similar triangles by pairwise connecting the midpoints of the three edges K. We call this family the red refinement of K and denote it by $\mathcal{S}K$.*

(ii) *Let $\mathcal{T}$ be a triangulation of a domain Ω, we say its red refinement is the mesh obtained applying the red refinement to each of its elements. We denote it by $\mathcal{ST}$.*

(iii) *Define recursively $\mathcal{S}^{(j+1)}\mathcal{T}$ as the red refinement of the grid $\mathcal{S}^{(j)}\mathcal{T}$, with $\mathcal{S}^{(1)}\mathcal{T} \overset{\mathbf{def}}{=} \mathcal{ST}$.*

Remark 3.5. (i) Notice that the size of the self-similar refinement is half the size of the original triangulation, *i.e.*, $h(\mathcal{ST}) = \dfrac{1}{2}\, h(\mathcal{T})$, see [**2**].

(ii) Since the red refinement of a mesh $\mathcal{S}(\mathcal{T})$ divides each triangle into four subtriangles, each of them similar to the original one, the regularity of the mesh remains equal *i.e.*, $\zeta(\mathcal{T}) = \zeta(\mathcal{ST}) = \zeta(\mathcal{S}^{(j)}\mathcal{T})$ for all $j \in \mathbb{N}$, see [**2**]. This is seen in Figure 2 which depicts two levels of red refinement for a given grid. The first one in dashed-doted line and a second level, executed only on the bottom triangle region, in doted line.

Theorem 3.25. *Let $\mathcal{T} = \{K \in \mathcal{T}\}$ be a bipartite triangulation of the domain Ω, then its red refinement $\mathcal{S}(\mathcal{T})$ is a bipartite grid.*

Proof. Let ξ be an interior vertex of the triangulation $\mathcal{ST}$, due to Theorem 3.20 we need to show that $\#\{L \in \mathcal{ST} : \xi \in \partial L\}$ is even. Notice that ξ has only two possibilities:

(i) ξ is an interior vertex of the original triangulation. Since the self-similar refinement does not introduce new edges incident on the vertices of the original triangulation $\mathcal{T}$ then

$$\#\{L \in \mathcal{ST} : \xi \in \partial L\} = \#\{K \in \mathcal{T} : \xi \in \partial K\}.$$

Since $\mathcal{T}$ is bipartite the cardinal of the right hand side is even.

(ii) There exists $\sigma \in \mathcal{E}_{\mathcal{T}}$, the set of edges of the original triangulation $\mathcal{T}$, such that ξ is the middle point of σ. Let $K_1, K_2 \in \mathcal{T}$ be the only pair such that $\sigma = \partial K_1 \cap \partial K_2$. Then, exactly six triangles of $\mathcal{ST}$ are incident on ξ: three contained in K_1 and three contained in K_2; i.e., $\#\{L \in \mathcal{ST} : \xi \in \partial L\}$ is even.

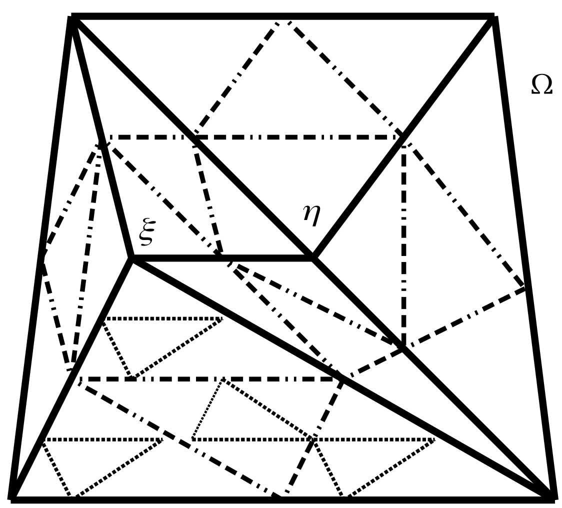

Figure 2. Red Refinement

Remark 3.6. It is important to observe that the red refinement does not generate bipartite grids. In Figure 2 two internal vertices of the original triangulation ξ and η are depicted. Notice that in both cases, the red refinement does not change the number of triangles incident on them. Therefore, it is necessary to use the refinement given in Definition 3.21 to generate bipartite grids; this is unfortunate due to the quality deterioration such procedure introduces in the mesh. However, in the practical case when bipartite grids of arbitrary small size are necessary, due to Theorem 3.25, the bipartite refinement needs to be applied only once. In particular the grid $\mathcal{S}^{(j)}\mathcal{BT}$ is bipartite and for $j \in \mathbb{N}$ large enough, the diameter of the elements will not be larger than the required bound of control.

4. THREE DIMENSIONAL CASE

We start this section by defining the basic topological spaces, and the main topological results, to analyze the bipartite tetrahedral grids problem.

Definition 4.26. *Let* $\mathcal{S}^1 \overset{\textbf{def}}{=} \{(x, y) \in \mathbb{R}^2 : |(x, y)| = 1\} \subseteq \mathbb{R}^2$ *and* $\jmath(\mathcal{S}^1) \overset{\textbf{def}}{=} \{(x, y, z) \in \mathbb{R}^3 : |(x, y)| = 1, z = 0\}$ *i.e., its "natural" embedding in* $\mathbb{R}^3$*. Define*

$$\mathcal{R} \overset{\textbf{def}}{=} \mathbb{R}^3 - \jmath(\mathcal{S}^1). \tag{3}$$

Theorem 4.27. *The space* $\mathcal{R}$ *is not simply connected.*

Proof. See Proposition 6.1 [**13**].

Proposition 4.28. *Let* $K \subseteq \mathbb{R}^3$ *be a tetrahedron,* Φ *one of its faces and* $\mathscr{S}$ *the contour of* Φ *i.e.,* $\mathscr{S} = \partial(\Phi)$ *in the trace topology of* Φ*. Then,*

(i) $\mathbb{R}^3 - \mathscr{S}$ *is homeomorphic to* $\mathbb{R}^3 - \jmath(\mathcal{S}^1) = \mathcal{R}$.

(ii) $\mathbb{R}^3 - \mathscr{S}$ *is not simply connected.*

Proof. (i) Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be any homeomorphism such that $h(\mathscr{S}) = \jmath(\mathcal{S}^1)$ then $h|_{\mathbb{R}^3 - \mathscr{S}}$ is a homeomorphism.

(ii) Since $\mathbb{R}^3 - \jmath(\mathcal{S}^1) = \mathcal{R}$ is not simply connected as shown in Theorem 4.27 and homeomorphic to $\mathbb{R}^3 - \mathscr{S}$ the result follows. ☐

Next, we are aimed to characterize bipartite tetrahedral grids. We start introducing the definitions in order to model a tetrahedral mesh with a graph.

Definition 4.29. *Let* $\mathcal{O} \subseteq \mathbb{R}^3$ *be an open bounded polyhedral domain and* $\{K : K \in \mathcal{T}\}$ *be a tetrahedral mesh of* $\mathcal{O}$*. We denote by* $\mathcal{P}$, $\mathcal{E}_{\mathcal{T}}$ *and* $\mathcal{F}_{\mathcal{T}}$ *the set of vertices, edges and faces of the tetrahedral mesh respectively.*

(i) *We say a face* $\Phi \in \mathcal{F}_{\mathcal{T}}$ *is EXTERIOR if it lies on the boundary of the domain* $\partial\mathcal{O}$ *(if* $|\Phi \cap \partial\mathcal{O}|_2 > 0$*) and denote the set of exterior faces by* $\mathcal{F}_{\text{ext}}$*. We say a face* $\Phi \in \mathcal{F}_{\mathcal{T}}$ *of a tetrahedron is INTERIOR if it belongs to the interior of the domain (if* $|\Phi \cap \partial\mathcal{O}|_2 = 0$*) and denote the set of interior faces by* $\mathcal{F}_{\text{int}}$*.*

(ii) *The associated graph* $\mathcal{G}_{\mathcal{T}}(\mathcal{T}, \mathcal{F}_{\text{int}})$ *is defined in the following way. The set of nodes* $\mathcal{T}$ *is defined by the set of tetrahedra, and there is an edge in* $\mathcal{F}_{\text{int}}$ *between two elements* $K, L \in \mathcal{T}$ *if* $K \neq L$ *and* $|\partial K \cap \partial L \cap \mathcal{O}|_2 > 0$ *i.e., if they are different elements and share a common face contained in the domain* $\mathcal{O}$*, except possibly for extreme subsets of the face* $\partial K \cap \partial L$*.*

(iii) *An element of the mesh* $\{K : K \in \mathcal{T}\}$ *is said to be isolated if its degree as node of* $\mathcal{G}_{\mathcal{T}}$ *is zero. We denote by* $\mathcal{T}_{\text{nct}}$ *the connected elements of the mesh.*

(iv) *We say a tetrahedron is INTERIOR if* $|\partial K \cap \partial\mathcal{O}|_2 = 0$ *(equivalently, if none of its faces is on the boundary of the domain), and denote by* $\mathcal{T}_{\text{int}}$ *the set of interior tetrahedra. We say a tetrahedron* $K \in \mathcal{T}$ *is EXTERIOR if* $|\partial K \cap \partial\mathcal{O}|_2 > 0$

(*equivalently, if one or more of its faces is on the boundary of the domain*), *and denote by* $\mathcal{T}_{\text{ext}}$ *the set of exterior tetrahedra.*

(v) *We say an edge* $\sigma \in \mathcal{E}_{\mathcal{T}}$ *is EXTERIOR if it lies on the boundary of the domain* $\partial\mathcal{O}$, *and denote the set of exterior edges by* $\mathcal{E}_{\text{ext}}$. *We say an edge* $\sigma \in \mathcal{E}_{\mathcal{T}}$ *of a tetrahedron is INTERIOR if it belongs to the interior of the domain, and denote the set of interior edges by* $\mathcal{E}_{\text{int}}$.

Definition 4.30. *Let* $\mathcal{O} \subseteq \mathbb{R}^3$ *be an open bounded polyhedral domain and* $\{K : K \in \mathcal{T}\}$ *a tetrahedral mesh of* $\mathcal{O}$. *We say a tetrahedral mesh* $\{K : K \in \mathcal{T}\}$ *is BIPARTITE if its associated graph* $\mathcal{G}_{\mathcal{T}}(\mathcal{T}, \mathcal{F}_{\text{int}})$ *is bipartite.*

Definition 4.31. *Let* $\mathcal{O} \subseteq \mathbb{R}^3$ *be an open bounded polygonal domain and* $\{K : K \in \mathcal{T}\}$ *be a tetrahedral mesh of* $\mathcal{O}$.

(i) *For each* $K \in \mathcal{T}$ *we denote by* $\mathbf{b}_K$ *its centroid or center of gravity.*

(ii) *Given a cycle* $C = \langle K_1, K_2, \ldots, K_j, K_1 \rangle$ *in the graph* $\mathcal{G}_{\mathcal{T}}$ *define its associated loop* γ_C, *by the sequence of segments* $[\mathbf{b}_{K_1}, \mathbf{b}_{K_2}], [\mathbf{b}_{K_2}, \mathbf{b}_{K_3}] \ldots [\mathbf{b}_{K_j}, \mathbf{b}_{K_1}]$ i.e., γ_C *is contained in* $\mathcal{O}$.

We are aimed to give necessary and sufficient conditions for a tetrahedral mesh of a domain Ω to be bipartite. First we focus on a very particular type of tetrahedral mesh.

Definition 4.32. *We say a tetrahedral mesh* $\{K : K \in \mathcal{T}\}$ *of an open, bounded, simply connected, polyhedral domain* $\mathcal{O}$ *is a TENT, if it has only one interior edge named POLE, denoted by* σ_p, *such that* $\{K \in \mathcal{T} : \sigma_p \subseteq \partial K\} = \mathcal{T}$. *We denote its associated graph by* $\mathcal{G}_{\mathcal{T}} = \mathcal{G}_{\sigma_p}$.

Remark 4.7. Notice that since the domain $\mathcal{O}$ is simply connected, the graph of the tetrahedral mesh $\mathcal{G}_{\mathcal{T}}$, must be connected.

Remark 4.8. (i) The Figure 3 displays the most basic case of a tent for a polyhedral (tetrahedral in this case) domain Ω. A polygonal domain $\omega \subseteq \mathbb{R}^2$ is subdivided in three triangles T_1, T_2, T_3 with only one interior vertex V. The pole of the mesh σ_p stands on the vertex V, "lifting" each triangle into a tetrahedron.

(ii) A more general type of tent consists on a polygonal domain triangulated with a radial triangulation (seen in Definition 3.15) i.e., all the triangles are incident in one single vertex V and the pole σ_p stands on the vertex, "lifting" each triangle into a tetrahedron.

(iii) In the most general version of a tent, there may not exist a plane hosting one face of each tetrahedron of the mesh.

Proposition 4.33. *Let* $\{K : K \in \mathcal{T}\}$ *be a tent mesh of the domain* Ω. *Then, the following statements hold.*

(i) *The degree of each vertex* K *in the graph of the mesh* $\mathcal{G}_{\mathcal{T}}$ *is* 2.

(ii) *A tent mesh has at least three elements.*

(iii) *If* $\#\mathcal{T} = j$ *then* $\mathcal{G}_{\mathcal{T}} = C_j$ i.e., *it is a cycle graph.*

(iv) *The tent mesh is bipartite if and only if the number of tetrahedra is even.*

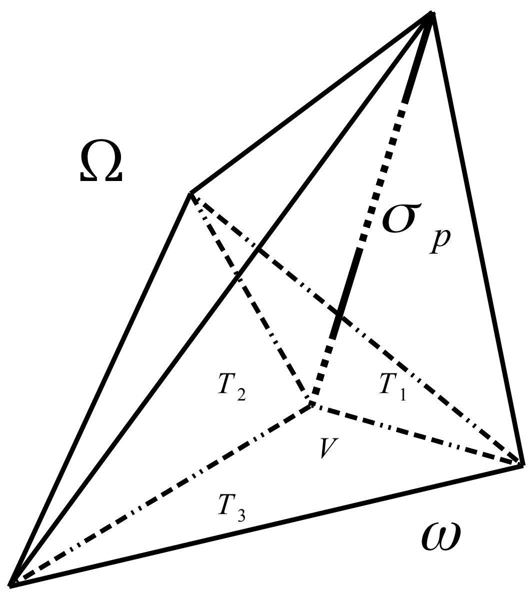

Figure 3. Tent–type Tetrahedral Mesh

Proof. (i) Let σ_p be the pole of the tent and K be an element of the mesh. Since $\sigma_p \in \partial K$ and only two faces of K host σ_p, this implies that the other two faces of K must be exterior, thus $\deg(K) \leq 2$. Since $\mathcal{G}_{\mathcal{T}}$ is connected we know $\deg(K) \geq 1$. However if $\deg(K) = 1$, this would imply that only one face of the tetrahedron is in the interior of Ω and the three faces of K would be exterior, which can not be since σ_p is an interior edge; this proves the first part.

(ii) If a mesh has less than three tetrahedra, i.e., two or one, there exists at most one interior face. Therefore, no interior edges exist and, by definition, the mesh can not be a tent.

(iii) The fact that $\deg(K) = 2$ for all $K \in \mathcal{T}$ implies $\#\mathcal{T} = \#\mathcal{F}_{\text{int}}$. On the other hand, by construction $\mathcal{G}_{\mathcal{T}}$ is simple, therefore it must be the cycle graph C_j, where j is the number of elements of $\mathcal{T}$.

(iv) Since $\mathcal{G}_{\mathcal{T}}$ is a cycle graph, it contains a unique cycle of length $\#\mathcal{T}$, then due to Theorem 2.7, it is bipartite if and only if $\#\mathcal{T}$ is even.

Definition 4.34. *Let $\mathcal{O} \subseteq \mathbb{R}^3$ be an open, bounded, simply connected, polyhedral domain, $\{K : K \in \mathcal{T}\}$ be a tetrahedral mesh of $\mathcal{O}$ and σ be an interior edge. We define its associated subgraph C_σ by the set of tetrahedra $V_\sigma \overset{\textbf{def}}{=} \{K \in \mathcal{T} : \sigma \in \partial K\}$ and the set of faces $\mathcal{F}_\sigma$ connecting two elements of V_σ. Clearly, C_σ is a tent and due to Proposition* 3.16, *we know that $C_\sigma = C_j$, where $j \overset{\textbf{def}}{=} \#V_\sigma$.*

Lemma 4.35. *Let $\{K : K \in \mathcal{T}\}$ be a tetrahedral mesh of Ω then, for any cycle C in the graph $\mathcal{G}_{\mathcal{T}}$ and for each $K \in C$, there exists at least one interior edge $\sigma \in \mathcal{E}_{\text{int}}$*

such that $\sigma \subseteq \partial K$.

Proof. Let C be a cycle in the graph $\mathcal{G}_\mathcal{T}$ and $\gamma_C : [0,1] \to \Omega$ be its associated loop. Without loss of generality, it can be assumed that its parametrization is bijective. Fix $K \in C$, and let $L_1, L_2 \in C$ be the elements such that the sequence $\langle L_1, K, L_2 \rangle$ is in the cycle. Let $\Phi \overset{\mathbf{def}}{=} \partial L_1 \cap \partial K$ i.e., the unique face shared by L_1 and K, since $|\gamma_C \cap L_1| > 0$ and $|\gamma_C \cap K| > 0$ the loop develops across the face Φ. On one hand, since γ_C joins the centroids with straight lines, it hits Φ. On the other hand, since the parametrization of γ_C is bijective the hit occurs only once, i.e., $\gamma_C(t_0) \in \Phi$ for a unique $t_0 \in (0,1)$. Let $\mathscr{S}$ be the contour of the face Φ i.e.,

$$\mathscr{S} \overset{\mathbf{def}}{=} \operatorname{cl} \bigcup_{\sigma \in \mathcal{E}_\mathcal{T}} \{\sigma : \sigma \subseteq \partial K \cap \partial L_1\}.$$

Then, $\mathscr{S}$ is a one-dimensional manifold in $\mathbb{R}^3$ with the shape of a triangle. If K has no interior edges it would hold that $\mathscr{S} \cap \Omega = \emptyset$, this would imply that Ω and consequently γ_C are contained in $\mathbb{R}^3 - \mathscr{S}$. However, the loop γ_C develops across Φ and hits this face only once. Since $\mathbb{R}^3 - \mathscr{S}$ is not simply connected, as seen in Proposition 4.28, γ_C can not be homotopic to one point. This contradicts the hypothesis for Ω being simply connected. Therefore, one of the edges σ of K must be interior.

Theorem 4.36. *Let* $\{K : K \in \mathcal{T}\}$ *be a tetrahedral mesh of* Ω *such that it has only one interior edge, namely* $\mathcal{E}_{\text{int}} = \{\sigma_p\}$. *Then, the mesh is bipartite if and only if* $\#\{K \in \mathcal{T} : \sigma_p \subseteq \partial K\}$ *is even.*

Proof. First observe that the mesh has at least three tetrahedra, otherwise no edge could be interior. If $\{K \in \mathcal{T} : \sigma_p \in \partial K\} = \mathcal{T}$ there is nothing to prove due to Proposition 4.33.

If $\mathcal{T} - \{K \in \mathcal{T} : \sigma_p \in \partial K\} \neq \emptyset$ consider the tent subgraph C_{σ_p} given by Definition 4.34, which is cyclic i.e., $C_{\sigma_p} = C_j$ where $j \overset{\mathbf{def}}{=} \#\{K \in \mathcal{T} : \sigma_p \in \partial K\}$.

Let L be in $\mathcal{T} - \{K \in \mathcal{T} : \sigma_p \in \partial K\}$ then, it must hold that its six edges are exterior and due to Lemma 4.35, L can not belong to any cycle. Hence, any cycle C in the graph $\mathcal{G}_\mathcal{T}$ must lie in the subgraph C_{σ_p}, however the subgraph $\mathcal{G}_{\sigma_p} = C_j$ is cyclic and its unique cycle is itself. Therefore, the graph $\mathcal{G}_\mathcal{T}$ contains a unique cycle and due to Theorem 2.7, it is bipartite if and only if $j = \#\{K \in \mathcal{T} : \sigma_p \in \partial K\}$ is even. □

Before characterizing bipartite grids, we need an intermediate lemma regarding the process of "removing" an exterior tetrahedron from a tetrahedral mesh.

Lemma 4.37. *Let* $\{L : L \in \mathcal{T}\}$ *be a tetrahedral mesh of* Ω, *and* K *be an exterior tetrahedron of* $\mathcal{T}$, *such that it has at least one interior edge* σ. *Define* $\mathcal{T}_K \overset{\mathbf{def}}{=} \{L \in \mathcal{T} : L \neq K\}$ *and the domain* $\Omega_K \overset{\mathbf{def}}{=} \Omega - \operatorname{cl}(K)$.

(i) *Then, the degree* $\deg(K) \in \{2,3\}$ *and* K *has at most three interior edges.*

(ii) *If* $\deg(K) = 2$, *then the graph* $\mathcal{G}_{\mathcal{T}_K}$ *is connected and consequently the domain* Ω_K *is also connected.*

(iii) *If* $\deg(K) = 3$ *and* K *has two or three interior edges then, the graph* $\mathcal{G}_{\mathcal{T}_K}$ *is connected and consequently the domain* Ω_K *is also connected.*

(iv) *If* $\deg(K) = 2$ *or* $\deg(K) = 3$ *and it has two or three interior edges, the domain* Ω_K *is simply connected.*

(v) *If* $\deg(K) = 2$ *or* $\deg(K) = 3$ *and it has only one interior edge, the domain* Ω_K *can have one or two components. However, each component of* Ω_K *is simply connected.*

Proof. (i) K has at least one interior edge, therefore its degree has to be greater than one. Additionally K is exterior, then its degree has to be less than 4 i.e., $\deg(K) \in \{2, 3\}$. Seeing that K is exterior, it has a face Φ contained on the boundary of Ω, therefore the three edges belonging to $\partial\Phi \cap \partial K$ have to be exterior, i.e., the element has at most three interior edges.

(ii) Let $L_1, L_2 \in \mathcal{T}$ be the only two tetrahedra such that $|\partial L_i \cap \partial K|_2 > 0$, $i = 1, 2$. It also holds that $\sigma \subseteq \partial L_1 \cap \partial L_2$ and by definition $L_1, L_2 \in \mathcal{T}_K$. Since $\sigma \in \mathcal{E}_{\text{int}}$, consider the tent subgraph C_σ which is contained in $\mathcal{G}_{\mathcal{T}}$. Clearly $L_1, L_2 \in C_\sigma$ and there exists a unique path $\langle L_1, \ldots, L_2\rangle$, such that it is contained in the cycle graph C_σ but does not hit K. Observing that all the elements of the path belong to $\mathcal{G}_{\mathcal{T}_K}$, we conclude that L_1 and L_2 can be connected in $\mathcal{G}_{\mathcal{T}_K}$, therefore this graph is connected. The connectedness of Ω_K follows immediately.

(iii) Let $L_1, L_2, L_3 \in \mathcal{T}$ be the only three tetrahedra such that $|\partial L_i \cap \partial K|_2 > 0$, $i = 1, 2, 3$, and let σ, τ be two interior edges of K. Without loss of generality assume that

$$\sigma \subseteq \partial L_1 \cap \partial L_2 \,, \quad \tau \subseteq \partial L_2 \cap \partial L_3.$$

By definition $L_1, L_2, L_3 \in \mathcal{T}_K$. Repeating the previous argument for C_σ, C_τ, we conclude that L_1, L_2 can be connected and L_2, L_3 can be connected. Consequently L_1 and L_3 can also be connected. Then, the graph $\mathcal{G}_{\mathcal{T}_K}$ is connected. From here the connectedness of Ω_K follows.

(iv) The domain Ω_K is connected as seen in (ii) and (iii), recalling that Ω is simply connected and that K is convex and exterior, we conclude $\Omega - \text{cl}(K)$ must also be simply connected.

(v) If $\deg(K) = 2$ let L_1, L_2 be the tetrahedra connected to K. Then, it should hold that $\sigma \subseteq \partial L_1 \cap \partial L_2$. From the second part we know that $\mathcal{G}_{\mathcal{T}_K}$ is connected. Additionally, we know that Ω is simply connected and that K is convex and exterior, then we conclude that $\Omega - \text{cl}(K)$ is simply connected.

If $\deg(K) = 3$ let L_1, L_2, L_3 be the tetrahedra neighboring K, then without loss of generality we can assume that $\sigma \subseteq \partial L_1 \cap \partial L_2$, and due to the previous analysis L_1, L_2 can be connected. Hence, Ω_K has at most two components. If Ω_K is connected it must be simply connected since K is convex, exterior and Ω is simply connected. If Ω_K is not connected then L_1, L_2 belong to one component, namely Θ_1, which is simply connected by the previous argument, and L_3 belongs to the other component, namely Θ_2. Thus, K is convex and exterior to the polyhedral domain $\operatorname{int}[\Theta_2 \cup \operatorname{cl}(K)] \subseteq \Omega$. Recalling that Ω is simply connected, we conclude that Θ_2 is simply connected. □

Now we can characterize bipartite grids in the following result.

Theorem 4.38. *Let $\{K : K \in \mathcal{T}\}$ be a tetrahedral mesh of Ω. The grid is bipartite if and only if, for every interior edge $\sigma \in \mathcal{E}_{\text{int}}$ the number of incident tetrahedra $\#\{K \in \mathcal{T} : \sigma \subseteq \partial K\}$ is even.*

Proof. We begin proving the necessity. Suppose that there exists an interior edge, namely $\sigma_p \in \mathcal{E}_{\text{int}}$, such that $\#\{K \in \mathcal{T} : \sigma_p \subseteq \partial K\} = 2j + 1$. Considering the tent subgraph $\mathcal{G}_{\sigma_p}$ and its unique cycle $C = \langle K_1, K_2, \ldots, K_{2j+1}, K_1\rangle$, we conclude that the mesh can not be bipartite due to Theorem 2.7.

In order to prove the sufficiency of the condition, we proceed by induction on the number of interior edges. The case $\#\mathcal{E}_{\text{int}} = 0$ implies that the graph $\mathcal{G}_{\mathcal{T}}$ has no cycles due to Lemma 4.35, and therefore it is bipartite according to Theorem 2.7. If $\#\mathcal{E}_{\text{int}} = 1$ the result follows due to Theorem 4.36. Assume now that the result holds whenever the number of interior edges is less or equal than j. Define

$$\mathcal{T}_{SC} \overset{\textbf{def}}{=} \{K \in \mathcal{T} : \#(\partial K \cap \mathcal{E}_{\text{ext}}) < 6\},$$

and

$$\Omega_{SC} \overset{\textbf{def}}{=} \bigcup\{K : K \in \mathcal{T}_{SC}\}.$$

i.e., the tetrahedra of the mesh which do not have six exterior edges, and Ω_{SC} the subdomain of Ω for which $\mathcal{T}_{SC}$ is a natural tetrahedral mesh. Clearly, the number of interior edges in $\mathcal{T}_{SC}$ equals the number of interior edges in $\mathcal{T}$ i.e., $j + 1$. Moreover, if σ is an interior edge of the tetrahedral mesh then

$$\#\{K \in \mathcal{T}_{SC} : \sigma \subseteq \partial K\} = \#\{K \in \mathcal{T} : \sigma \subseteq \partial K\}.$$

Recall that if $\#(\partial K \cap \mathcal{E}_{\text{ext}}) = 6$ then, due to Lemma 4.35, K can not belong to any cycle. Consequently the cycles in $\mathcal{T}_{SC}$ and in $\mathcal{T}$ are the same. Moreover, a cycle is even in $\mathcal{T}_{SC}$ if and only if it is even in $\mathcal{T}$. Hence, without loss of generality, it can be assumed that the tetrahedral grid satisfies $\#(\partial K \cap \mathcal{E}_{\text{ext}}) < 6$ for all $K \in \mathcal{T}$.

Let K be an exterior tetrahedron, then one of its edges, namely σ, must be interior. From now on we denote by K_σ this element of $\mathcal{T}$. Consider the domain $\Omega_\sigma \overset{\textbf{def}}{=} \Omega - \operatorname{cl}(K_\sigma)$, by definition $\sigma \nsubseteq \Omega_\sigma$. On the other hand, the family $\mathcal{T}_\sigma \overset{\textbf{def}}{=} \{K \in \mathcal{T} : K \neq K_\sigma\}$ is clearly a tetrahedral mesh of the domain Ω_σ, where σ is not an interior edge. Since K_σ is exterior but has an interior edge then, due to Lemma 4.37, its degree is two or three; and it could have one, two or three interior edges, one of which must be σ. In any of the cases, the set of interior edges $\mathcal{E}_{\text{int}}(\mathcal{T}_\sigma)$ of the triangulation $\mathcal{T}_\sigma$ is contained in $\mathcal{E}_{\text{int}} - \{\sigma\}$ i.e., it has at most j interior edges. Moreover, the fact that only one exterior tetrahedron was removed from $\mathcal{T}$ yields

$$\#\{L \in \mathcal{T}_\sigma : \tau \subseteq \partial L\} = \#\{L \in \mathcal{T} : \tau \subseteq \partial L\}, \quad \forall\, \tau \in \mathcal{E}_{\text{int}}(\mathcal{T}_\sigma).$$

Therefore, due to the hypothesis, each interior edge of the tetrahedral mesh $\mathcal{T}_\sigma$ has an even number of incident tetrahedra. Before the induction hypothesis can be applied, several cases have to be analyzed.

Case A. $\deg(K_\sigma) = 2$ or $\deg(K_\sigma) = 3$, with $\#(\mathcal{E}_{\text{int}} \cap \partial K_\sigma) = 2$ or $\deg(K_\sigma) = 3$, with $\#(\mathcal{E}_{\text{int}} \cap \partial K_\sigma) = 3$. In this case, due to Lemma 4.37, the domain Ω_σ is simply connected and since it has at most j interior edges, the induction hypothesis implies that the graph $\mathcal{G}_{\mathcal{T}_\sigma}$ is bipartite i.e., there exists a node bipartition of the graph U_σ, W_σ. Next, we analyze all the possible subcases.

(i) $\deg(K_\sigma) = 2$. This implies $\#(\mathcal{E}_{\text{int}} \cap \partial K_\sigma) = \#\{\sigma\} = 1$. Consider the tent subgraph C_σ given by Definition 4.34. From the hypothesis we know that C_σ contains an even number of tetrahedra, then the set $\{L \in \mathcal{T}_\sigma : \sigma \subseteq \partial L\}$ has an odd number of tetrahedra. Let $L_1, L_2 \in \mathcal{T}$ be the two tetrahedra such that $|\partial K_\sigma \cap \partial L_i|_2 > 0$ and $\sigma \subseteq \partial L_i$ for $i = 1, 2$. By definition $L_1, L_2 \in \mathcal{T}_\sigma$, we claim that these tetrahedra belong to only one subset of the node bipartition. Let $\langle L_1, \ldots, L_2 \rangle$ be the unique path from L_1 to L_2 within both graphs, C_σ and $\mathcal{G}_{\mathcal{T}_\sigma}$; clearly it has even length $\#C_\sigma - 2$. Then, since the graph $\mathcal{G}_{\mathcal{T}_\sigma}$ is bipartite, L_1 and L_2 belong to the same element of the node partition, either U_σ or W_σ. Without loss of generality assume that $L_1, L_2 \in U_\sigma$. Thus, the pair $U \overset{\textbf{def}}{=} U_\sigma$, $W \overset{\textbf{def}}{=} W_\sigma \cup \{K_\sigma\}$ is a node bipartition of the graph $\mathcal{G}_{\mathcal{T}}$ because it has only one extra element, K_σ, whose only two edges have the other endpoint on a tetrahedron belonging to U. The proof is complete for this case.

(ii) $\deg(K_\sigma) = 3$, with $\#(\mathcal{E}_{\text{int}} \cap \partial K_\sigma) = \#\{\sigma, \tau\} = 2$. Let L_1, L_2, L_3 be the tetrahedra such that $|\partial L_i \cap \partial K_\sigma|_2 > 0$ for $i = 1, 2, 3$, without loss of generality we can assume that,

$$\sigma \subseteq \partial L_1 \cap \partial L_2\,, \quad \tau \subseteq \partial L_2 \cap \partial L_3. \tag{5}$$

Notice that L_1, L_2 and L_3 belong to the graph $\mathcal{G}_{\mathcal{T}_\sigma}$. Let C_σ and C_τ be the tent subgraphs given by Definition 4.34, then they both have an even number of tetrahedra. Also, due to (5) $L_1, L_2 \in C_\sigma$ and $L_2, L_3 \in C_\tau$ must hold. Let $\langle L_1, \ldots, L_2 \rangle$ be the unique path from L_1 to L_2 within both graphs, C_σ and $\mathcal{G}_{\mathcal{T}_\sigma}$, clearly it has even length $\#C_\sigma - 2$. Then, since the graph $\mathcal{G}_{\mathcal{T}_\sigma}$ is bipartite, L_1 and L_2 must belong to the same subset of the node partition, either U_σ or W_σ. Repeating the argument, L_2 and L_3 must belong to the same subset either U_σ or W_σ, therefore we conclude that the three of them belong to one single set. Without loss of generality, assume that $L_1, L_2, L_3 \in U_\sigma$. Hence, the pair $U \overset{\textbf{def}}{=} U_\sigma$, $W \overset{\textbf{def}}{=} W_\sigma \cup \{K_\sigma\}$ is a node bipartition of the graph $\mathcal{G}_{\mathcal{T}}$, since it only has one extra element K_σ, whose only three edges have the other endpoint on a tetrahedron belonging to U. The case has been proved.

(iii) $\deg(K_\sigma) = 3$, with $\#(\mathcal{E}_{\text{int}} \cap \partial K_\sigma) = \#\{\sigma, \tau, \varrho\} = 3$. Let L_1, L_2, L_3 be the tetrahedra such that $|\partial L_i \cap \partial K_\sigma|_2 > 0$ for $i = 1, 2, 3$, and such that

$$\sigma \subseteq \partial L_1 \cap \partial L_2\,, \quad \tau \subseteq \partial L_2 \cap \partial L_3\,, \quad \varrho \subseteq \partial L_3 \cap \partial L_1. \tag{6}$$

This case can be reduced to the previous one, A(ii), since it is enough to analyze the tent subgraphs C_σ and C_τ to conclude that L_1, L_2 and L_3 belong to the same subset of the bipartition U_σ of the graph $\mathcal{G}_{\mathcal{T}_\sigma}$. Using the same arguments as before, we have that the pair $U \stackrel{\textbf{def}}{=} U_\sigma$, $W \stackrel{\textbf{def}}{=} W_\sigma \cup \{K_\sigma\}$ is a node bipartition of the graph $\mathcal{G}_\mathcal{T}$.

Case B. $\boldsymbol{\deg(K_\sigma) = 3}$ **and** $\boldsymbol{\#(\mathcal{E}_{\text{int}} \cap \partial K_\sigma) = 1}$**.** Let L_1, L_2, L_3 be the three neighboring tetrahedra to K_σ and assume that $\sigma \subseteq \partial L_1 \cap \partial L_2$. In this case, due to Lemma 4.37(v), the domain Ω_σ has one or two connected components. If it has only one component the problem is addressed using the technique presented for the case A(i).

If Ω_σ has two components, namely Θ_1, Θ_2, due to Lemma 4.37(v), each of them is simply connected and $L_1, L_2 \in \Theta_1$, $L_3 \in \Theta_2$. Using the induction hypothesis, both graphs $\mathcal{G}_{\Theta_1}$ and $\mathcal{G}_{\Theta_2}$ are bipartite. Let (U_σ, W_σ) and (U', W') be the node bipartition pairs of both graphs, where $L_1, L_2 \in U_\sigma$ and $L_3 \in U'$. We prove that the pair $U \stackrel{\textbf{def}}{=} U_\sigma \cup U'$, $W \stackrel{\textbf{def}}{=} W_\sigma \cup \{K_\sigma\} \cup W'$ is a node bipartition for the graph $\mathcal{G}_\mathcal{T}$. Observe that if a closed path C hits elements on both subgraphs $\mathcal{G}_{\Theta_1}, \mathcal{G}_{\Theta_2}$, it can not be a cycle. Let C be a closed path, since the one element that connects the subgraphs $\mathcal{G}_{\Theta_1}, \mathcal{G}_{\Theta_2}$ is K_σ, then it must be part of the path C. However, because it is a closed path, it can not hit $\mathcal{G}_{\Theta_1}$ and $\mathcal{G}_{\Theta_2}$ without having to hit K_σ twice, therefore C can not be a cycle. Now let C be a cycle in the graph $\mathcal{G}_\mathcal{T}$, due to the previous discussion only two cases are possible.

(i) C belongs to the subgraph of the domain $\Theta_1 \cup K_\sigma$. This case can be reduced to A(i) since K_σ has degree two and only one interior edge in the subgraph corresponding to the mesh inherited to the domain $\Theta_1 \cup K_\sigma$. Therefore, the pair $U \stackrel{\textbf{def}}{=} U_\sigma$, $V \stackrel{\textbf{def}}{=} W_\sigma \cup \{K_\sigma\}$ constitutes a node bipartition for the graph of the subdomain $\Theta_1 \cup \{K_\sigma\}$. Then, the length of C must be even.

(ii) C belongs to the subgraph of the domain $\Theta_2 \cup K_\sigma$. In this case K_σ has no edges which are interior to the domain $\Theta_2 \cup K_\sigma$. Then, as seen in Lemma 4.35, the element K_σ can not belong to the cycle. Thus C must be contained in the graph of the domain Θ_2, which is bipartite i.e., its length must be even. Since in both subcases the length of the cycle is even, this proves the case B, and the proof of the theorem is complete. □

We close the section presenting a result for the existence of a bipartite tetrahedral grid, for simply connected polyhedral domains in $\mathbb{R}^3$. As in the two dimensional setting presented in the previous section, the process will deteriorate the quality of the grid, and here we estimate that deterioration. We start recalling some well-known properties of the tetrahedron.

Theorem 4.39. *Let Δ be a non-degenerate tetrahedron, and $\{\mathbf{v}_\ell : 1 \leq \ell \leq 4\}$ be its four vertices, with $\mathbf{v}_\ell = (x_\ell, y_\ell, z_\ell)$. Then, the center of gravity $\overline{\mathbf{v}} = (\overline{x}, \overline{y}, \overline{z})$ satisfies $\overline{\mathbf{v}} = \frac{1}{4}\sum_{\ell=1}^{4} \mathbf{v}_\ell$. Moreover, if $\mathbf{b} = (b_x, b_y, b_z)$ is the centroid of the face of the* *tetrahedron defined by the first three vertices* $\{\mathbf{v}_\ell : 1 \leq \ell \leq 3\}$, *then the points* $\mathbf{v}_4, \overline{\mathbf{v}}$ *and* $\mathbf{b}$ *are collinear.*

Recall now the standard definition of the symmetric group of a set of "n letters".

Definition 4.40. *Given* $n \in \mathbb{N}$, *we define* S_n *as the set of all possible permutations of the set* $\{1, 2, \ldots, n\}$.

Next, we define a convenient tetrahedral mesh for a non-degenerate tetrahedron

Definition 4.41. *Let* Δ *be a tetrahedron and* $\{\mathbf{v}_\ell : 1 \leq \ell \leq 4\}$ *be its vertices. For each* $\pi \in S_4$ *define*

$$\Delta_\pi \overset{\mathbf{def}}{=} \operatorname{co}\left\{ \frac{1}{i} \sum_{\ell=1}^{i} \mathbf{v}_{\pi(\ell)} : 1 \leq i \leq 4 \right\}, \tag{7}$$

where $\operatorname{co}(A)$ *denotes the convex hull of the set* A. *We define the family*

$$\mathcal{B}\Delta = \{\Delta_\pi : \pi \in S_4\}, \tag{8}$$

as the BIPARTITE GRIDDING of Δ.

Remark 4.9. Some observations are in order for the definition above.

(i) Notice that for any $\pi \in S_4$, the vertices of Δ_π are a vertex of Δ, the midpoint of an edge of Δ, the centroid of one face of Δ and finally the centroid of Δ.

(ii) For any Δ_π, three of its vertices are convex combinations of points belonging to extreme sets of Δ: a vertex, an edge and a face. On the other hand, since Δ is non-degenerate, these three vertices of Δ_π can not be collinear.

(iii) Due to Theorem 4.39, the fourth vertex of Δ_π is the centroid of Δ and since the original tetrahedron is non-degenerate, it can not be coplanar with the aforementioned three vertices of Δ_π.

(iv) Notice that $\#\mathcal{B}\Delta = \#S_4 = 24$.

Theorem 4.42. *Let* $\Delta \subseteq \mathbb{R}^3$ *be a non-degenerate tetrahedron and* $\{L_i : 1 \leq i \leq 24\}$ *be its bipartite gridding, then the associated graph is bipartite.*

Proof. Let $\{L_i : 1 \leq i \leq 24\}$ be the bipartite gridding of Δ. We classify the associated vertices in the following subsets, in terms of the original tetrahedron Δ.

(i) $\{\mathbf{v}_\ell : 1 \leq \ell \leq 4\}$ the vertices belonging to Δ.

(ii) ξ_Δ center of gravity of Δ.

(iii) $\{\mathbf{b}_\ell : 1 \leq \ell \leq 6\}$ the centroids of each face of Δ.

(iv) $\{\mathbf{m}_\ell : 1 \leq \ell \leq 6\}$ the midpoints of each of the edges of Δ. In the bipartite gridding $\{L_i : 1 \leq i \leq 24\}$, it is clear that an edge σ is interior if and only if one of its end points is the center of gravity ξ_{Δ}; this leaves three possible subcases:

a) The other end of σ is one of the vertices $\{\mathbf{v}_\ell : 1 \leq \ell \leq 6\}$ belonging to Δ. Then, six tetrahedra concur to it: two from each face concurrent to the vertex $\mathbf{v}_\ell$.

b) The other end of σ is one of the centroids $\{\mathbf{b}_\ell : 1 \leq \ell \leq 4\}$ of the faces of Δ, namely Φ. In such case, six tetrahedra concur to it, all of them having one of its faces contained in Φ.

c) The other end of σ is one of the midpoints $\{\mathbf{m}_\ell : 1 \leq \ell \leq 6\}$, of the edges of Δ. In this case four tetrahedra concur to it: two from each face concurrent to the edge that hosts $\mathbf{m}_\ell$.

Since in all the cases the number of tetrahedra concurrent to an interior edge is even, the result holds. □

Now we introduce a new definition.

Definition 4.43. *Let $\mathcal{T} = \{K : K \in \mathcal{T}\}$ be a tetrahedral mesh of the polyhedral domain Ω. We define its BIPARTITE REFINEMENT, as the mesh that is generated applying the bipartite gridding process to each tetrahedron K of the mesh. We denote it by $\mathcal{BT} = \{L : L \in \mathcal{BT}\}$.*

Theorem 4.44. *Let $\Omega \subseteq \mathbb{R}^3$ be an open, bounded, simply connected, polyhedral region. Then, there exists a tetrahedral mesh whose associated graph is bipartite.*

Proof. Let $\mathcal{T} = \{K : K \in \mathcal{T}\}$ be any tetrahedral mesh of Ω, since the domain is polyhedral such mesh exists. Denote by $\mathcal{BT} = \{L : L \in \mathcal{BT}\}$ its bipartite refinement. Let σ be an interior edge of the tetrahedral mesh $\mathcal{BT}$, due to Theorem 4.38, we need to prove that $\#\{L \in \mathcal{BT} : \sigma \in \partial L\}$ is even. Notice that σ has only three possibilities:

(i) If there exists $K \in \mathcal{T}$ such that $\sigma \subseteq \text{int}(K)$ then it is one of the edges of $\mathcal{B}K$. As seen in Theorem 4.42, every edge interior to one single tetrahedron generated by the bipartite gridding process, has an even number of tetrahedra incident on it.

(ii) If σ was part of an interior edge of the tetrahedral mesh $\mathcal{T}$, then $\#\{K \in \mathcal{T} : K \text{ is incident on } \sigma\} \neq 0$. For each tetrahedron $K \in \mathcal{T}$, incident on σ, the bipartite refinement $\mathcal{BT}$ generates two tetrahedra concurrent on σ. Therefore, the number of tetrahedra incident on σ belonging to $\mathcal{BT}$, is twice the number of tetrahedra incident on σ belonging to $\mathcal{T}$ i.e., such number is even.

(iii) If there exists $K \in \mathcal{T}$ such that the interior of one of its faces contains σ, then there must also exist another tetrahedron $L \in \mathcal{T}$ sharing the face that contains σ. Two tetrahedra concur to this edge from $\mathcal{B}K$ and other two from $\mathcal{B}L$, giving a total of four tetrahedra of $\mathcal{BT}$ incident on σ, which is an even number.

In the three cases above the number of tetrahedra concurrent on σ is even, then the result follows.

5. THE REFINEMENT

In this section we propose a method to generate bipartite grids of arbitrary small size and bounded regularity, i.e., the quality of the grids does not degenerate. We start this section with several definitions of geometrical shape parameters and then continue discussing their relationships.

Definition 5.45. *Let Δ be a non-degenerate tetrahedron in $\mathbb{R}^3$, define*

$$h_{\Delta} \overset{\mathbf{def}}{=} \textit{diameter of the tetrahedron } \Delta, \tag{9a}$$

$$\rho_{\Delta} \overset{\mathbf{def}}{=} \sup\{\textit{diameter of } B : B \textit{ is a ball contained in } \Delta\}. \tag{9b}$$

The regularity of the tetrahedron

$$\zeta_{\Delta} \overset{\mathbf{def}}{=} \frac{h_{\Delta}}{\rho_{\Delta}}. \tag{9c}$$

The radius ratio is defined by

$$\vartheta_{\Delta} \overset{\mathbf{def}}{=} 3\,\frac{r_{\text{in}}}{r_{\text{circ}}}, \tag{9d}$$

where $r_{\text{in}}, r_{\text{circ}}$ are respectively, the inradius and circumradius of Δ, see [**11**]. *Finally the mean ratio defined in* [**10**] *is given by*

$$\eta_{\Delta} \overset{\mathbf{def}}{=} \frac{12(3\,|\Delta|_3)^{2/3}}{\sum\{|\sigma|^2 : \sigma \textit{ is an edge of } \Delta\}}. \tag{9e}$$

Here $|\Delta|_3$ is the volume of the tetrahedron.

Recall now some previous results for relationship between shape parameters.

Theorem 5.46. *For any tetrahedron Δ it holds that*

$$\eta_{\Delta}^3 \leq \vartheta_{\Delta} \leq \frac{2}{\sqrt[4]{6}}\,\eta_{\Delta}^{3/4} \tag{10}$$

Furthermore, the lower bound is optimal and tight, and the upper bound is optimal.

Proof. See [**11**].

Remark 5.10. Let $c_0\,\eta_{\Delta}^{e_0} \leq \vartheta_{\Delta} \leq c_1\,\eta_{\Delta}^{e_1}$ be a tetrahedron shape measure estimate.

(i) If e_0 (or e_1) is the minimum (or maximum) possible exponent, then we say that the lower (or upper) bound is **optimal**.

(ii) If c_0 (or c_1) is the maximum (or minimum) possible constant, then we say that the lower (or upper) bound is **tight**.

It is obvious that

$$\zeta_\Delta = \frac{h_\Delta}{\rho_\Delta} = \frac{h_\Delta}{2\,r_{\rm in}} \le \frac{2\,r_{\rm circ}}{2\,r_{\rm in}} = \frac{3}{3\,\dfrac{r_{\rm in}}{r_{\rm circ}}}.$$

Where the inequality holds since the diameter of the tetrahedron is at most the diameter of the circumradius i.e., $h_\Delta \le 2\,r_{\rm circ}$. Then, $\zeta_\Delta \le \dfrac{3}{\vartheta_\Delta}$ and therefore

$$\zeta_\Delta \le \frac{3}{\eta_\Delta^3}. \tag{11}$$

Next, we study the deterioration of regularity when applying the bipartite refinement to a given tetrahedron.

Theorem 5.47. *Let K be a non-degenerate tetrahedron then, for any $L \in \mathcal{B}K$ holds*

$$\frac{1}{\eta_L} \le \frac{36\sqrt[3]{9}}{\eta_K}. \tag{12}$$

Proof. Fix $L \in \mathcal{B}K$; denote by $\{\mathbf{b}_i : 1 \le i \le 4\}$ its vertices and let $\{\mathbf{v}_i : 1 \le i \le 4\}$ be the vertices of K. We assume that the tetrahedron L, is the one generated by the identity permutation i.e., $L = \Delta_{Id}$ and its vertices satisfy

$$\mathbf{b}_i = \frac{1}{i}\sum_{\ell=1}^{i} \mathbf{v}_\ell\,, \quad 1 \le i \le 4. \tag{13}$$

Clearly, $\{|\mathbf{v}_i - \mathbf{v}_j| : 1 \le i < j \le 4\}$ and $\{|\mathbf{b}_i - \mathbf{b}_j| : 1 \le i < j \le 4\}$ are the lengths of the six edges of K and L respectively, due to (13) we have

$$|\mathbf{b}_1 - \mathbf{b}_2| = \left|\mathbf{v}_1 - \frac{\mathbf{v}_1 + \mathbf{v}_2}{2}\right| = \frac{1}{2}\,|\mathbf{v}_1 - \mathbf{v}_2| \le \frac{1}{2}\sum_{1\le i<j\le 4} |\mathbf{v}_i - \mathbf{v}_j|.$$

Exhausting the remaining five cases with the same technique we have:

$$|\mathbf{b}_k - \mathbf{b}_\ell| \le \frac{1}{\mathrm{lcm}\{k,\ell\}}\sum_{1\le i<j\le 4} |\mathbf{v}_i - \mathbf{v}_j| \le \frac{1}{2}\sum_{1\le i<j\le 4} |\mathbf{v}_i - \mathbf{v}_j|, \quad 1 \le k < \ell \le 4.$$

Here, $\mathrm{lcm}\{k,\ell\}$ denotes the lowest common multiple of k and ℓ. Therefore,

$$\begin{aligned}\Big(\sum_{1\le i<j\le 4} |\mathbf{b}_i - \mathbf{b}_j|^2\Big)^{1/2} &\le \sqrt{6}\,\max\{|\mathbf{b}_i - \mathbf{b}_j| : 1 \le i < j \le 4\} \\ &\le \frac{\sqrt{6}}{2}\sum_{1\le i<j\le 4} |\mathbf{v}_i - \mathbf{v}_j| \le \frac{\sqrt{6}}{2}\,\sqrt{6}\Big(\sum_{1\le i<j\le 4} |\mathbf{v}_i - \mathbf{v}_j|^2\Big)^{1/2}.\end{aligned}$$

Where the factor $\sqrt{6}$ shows due to the equivalence norms $\|\cdot\|_\infty - \|\cdot\|_2$ and $\|\cdot\|_1 - \|\cdot\|_2$ norms in $\mathbb{R}^6$. Thus,

$$\sum_{1 \leq i < j \leq 4} |\mathbf{b}_i - \mathbf{b}_j|^2 \leq 9 \sum_{1 \leq i < j \leq 4} |\mathbf{v}_i - \mathbf{v}_j|^2. \tag{14}$$

On the other hand, since the refinement $\mathcal{B}K = \{L_j : 1 \leq j \leq 24\}$ is made through the centroid of K, all the tetrahedra have the same volume i.e., $|L|_3 = \dfrac{1}{24}\,|K|_3$. Thus, recalling the definition of mean ratio given by equality (9e), and combining it with the inequality (14) above, we get

$$\begin{aligned}\eta_L = \frac{12(3|L|_3)^{2/3}}{\sum\{|\mathbf{b}_i - \mathbf{b}_j|^2 : 1 \leq i < j \leq 4\}} &= \frac{12(3\,\frac{1}{24}|K|_3)^{2/3}}{\sum\{|\mathbf{b}_i - \mathbf{b}_j|^2 : 1 \leq i < j \leq 4\}} \\ &\geq \frac{1}{9}\,\frac{1}{24^{2/3}}\,\frac{12(3|K|_3)^{2/3}}{\sum\{|\mathbf{v}_i - \mathbf{v}_j|^2 : 1 \leq i < j \leq 4\}} = \frac{1}{36\sqrt[3]{9}}\,\eta_K.\end{aligned}$$

This is the desired estimate (12).

Finally, for any other tetrahedron $M \in \mathcal{B}K$, let $\{\mathbf{u}_i : 1 \leq i \leq 4\}$ be its vertices, then there exists $\pi \in S_4$ such that

$$\mathbf{u}_i = \frac{1}{i}\sum_{\ell = 1}^{i} \mathbf{v}_{\pi(\ell)}\,, \quad \forall\, 1 \leq i \leq 4.$$

Repeating the previous arguments we have

$$\sum_{1 \leq i < j \leq 4} |\mathbf{u}_i - \mathbf{u}_j|^2 \leq 9 \sum_{1 \leq i < j \leq 4} |\mathbf{v}_{\pi(i)} - \mathbf{v}_{\pi(j)}|^2 = 9 \sum_{1 \leq i < j \leq 4} |\mathbf{v}_i - \mathbf{v}_j|^2,$$

and the estimate (12) follows.

Corollary 5.48. *Let K be a non-degenerate tetrahedron then, for any $L \in \mathcal{B}K$ holds*

$$\zeta_L \leq \frac{2^6 \cdot 3^9}{\eta_K^3}. \tag{15}$$

Proof. A straightforward combination of inequalities (11) and (15). □

We close this section citing a result given by the QLRS (quality local refinement based on subdivision) algorithm, which is described in [**12**].

Theorem 5.49. *Let $\mathcal{T}$ be a tetrahedral mesh of the domain Ω. Let $L^{(n)}$ be a refined tetrahedron produced by QLRS, where n denotes the number of refinement levels. Let $K \in \mathcal{T}$ be the tetrahedron such that $L^{(n)} \subseteq K$, then*

$$\eta(L^{(n)}) \geq \frac{\sqrt[3]{4}}{11}\,\eta(K). \tag{16}$$

Where $\eta(L^{(n)})$ and $\eta(K)$ stand for the mean ratio shape parameters of $L^{(n)}$ and K respectively.

REMARK 5.11. Notice that the QLRS algorithm can be applied as many times as needed, keeping a positive lower bound for its mean ratio, regardless of the number times it is applied. Therefore, using an original tetrahedral mesh, this can be refined in order to generate a tetrahedral grid of prescribed size $h > 0$, and its tetrahedra have mean ratio bigger or equal than $\frac{\sqrt[3]{4}}{11}\,\eta$; where $\eta \stackrel{\mathbf{def}}{=} \min\{\eta_\Delta : \Delta \in \mathcal{T}\} > 0$.

Definition 5.50. *Let $\{K : K \in \mathcal{T}\}$ be a tetrahedral mesh of the domain Ω, we denote by $\mathcal{Q}^n\mathcal{T}$ the n levels of QLRS refinement of the mesh $\mathcal{T}$.*

Theorem 5.51. *Let $\mathcal{T}$ be a tetrahedral mesh of Ω, then the sequence $\{\mathcal{B}\mathcal{Q}^n T : n \in \mathbb{N}\}$ has bounded regularity. Moreover, there exists a positive constant κ such that*

$$\sup_{n \in \mathbb{N}} \max\{\zeta_L : L \in \mathcal{B}\mathcal{Q}^n T\} \leq \frac{\kappa}{\eta^3}. \tag{17}$$

Where $\eta \stackrel{\mathbf{def}}{=} \min\{\eta_\Delta : \Delta \in \mathcal{T}\}$.

Proof. Let L be a tetrahedron in $\mathcal{B}\mathcal{Q}^n\mathcal{T}$, then there exists $K \in \mathcal{Q}^n\mathcal{T}$ and $\Delta \in \mathcal{T}$ such that $L \subseteq K \subseteq \Delta$. Hence, combining inequalities (15) and (16) we have

$$\zeta_L \leq \frac{3^6}{\eta_K^3} \leq \frac{11 \cdot 2^6 \cdot 3^9}{\sqrt[3]{4}} \frac{1}{\eta_\Delta^3} \leq \frac{11 \cdot 2^6 \cdot 3^9}{\sqrt[3]{4}} \frac{1}{\min\{\eta_\Delta^3 : \Delta \in \mathcal{T}\}}.$$

Therefore, the result follows for $\kappa = \frac{11 \cdot 2^6 \cdot 3^9}{\sqrt[3]{4}}$.

6. CONCLUDING REMARKS AND DISCUSSION

The present work yields several conclusions listed below.

(i) We have provided a method for generating bipartite grids of prescribed size and controlled regularity in 2-D and 3-D. This provides the theoretical setting to assure well-posedness for the mixed-mixed formulation of a problem, such as the porous media equation, presented in [**14**].

(ii) The method is far from being optimal. The bipartite refinements given in Definitions 3.21 and 4.43 for 2-D and 3-D respectively, subdivide internal angles of the initial elements. This deteriorates the shape quality of the mesh severely, as seen in the proof of Theorem 5.51, where the bound provided by the QLRS refinement method is amplified by a factor of $2^6 \cdot 3^9$.

(iii) One initial improvement for the bipartite refinements would discuss the placement of the new internal vertex. For instance in 2-D the centroid of the triangle could be replaced by the incenter, in order to bisect the angles of the original triangle. This deteriorates the internal angles in a more balanced way.

(iv) The bipartite refinement needs to be applied to the whole mesh, regardless of a-posteriori estimation of the solution or other guideline. This constraint needs to be addressed: from the mesh generation point of view (e.g. local refinement criteria or developing a different generation technique), and from the mixed-mixed variational formulation [**14**] point of view.

(v) Using the bipartite refinement generates a grid which has 6 times and 24 times, the number of elements of the original grid for 2-D and 3-D respectively; therefore the number of elements increases considerably. However, the computational costs for finding the center of gravity and code implementation are low, in contrast with the costs of calculating an optimal point, such as the incenter previously suggested.

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