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DICHOTOMY AND POSITIVITY OF NEUTRAL EQUATIONS WITH NONAUTONOMOUS PAST

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Consider the linear partial neutral functional differential equations with nonautonomous past of the form

$$\frac{\partial}{\partial t}F(u(t,\cdot)) = BFu(t,\cdot) + \Phi u(t,\cdot), \qquad t \geq 0,$$
$$\frac{\partial}{\partial t}u(t,s) = \frac{\partial}{\partial s}u(t,s) + A(s)u(t,s), \qquad t \geq 0 \geq s,$$

where the function $u(\cdot\,,\cdot)$ takes values in a Banach space X. Under appropriate conditions on the difference operator F and the delay operator Φ we prove that the solution semigroup for this system of equations is hyperbolic (or admits an exponential dichotomy) provided that the backward evolution family $\mathcal{U} = (U(t,s))_{t\leq s\leq 0}$ generated by $A(s)$ is uniformly exponentially stable and the operator B generates a hyperbolic semigroup $(e^{tB})_{t\geq 0}$ on X. Furthermore, under the positivity conditions on $(e^{tB})_{t\geq 0}$, $\mathcal{U}$, F and Φ we prove that the above-mentioned solution semigroup is positive and then show a sufficient condition for the exponential stability of this solution semigroup.

1. INTRODUCTION

Linear partial neutral functional differential equations with infinite difference and delay can be formulated in abstract forms as

$$\begin{cases} \dfrac{\partial}{\partial t}Fu_t = BFu_t + \Phi u_t & \text{for } t \geq 0, \\ u_0(t) \quad = \varphi(t) & \text{for } t \leq 0, \end{cases} \tag{1.1}$$

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where $u(\cdot)$ takes values in a Banach space X, B is a linear operator on X (representing the partial differential operator), while F and Φ are called *difference operator* and *delay operator* (respectively) which are bounded, linear operators from an X-valued function space, e.g., $C_0(\mathbb{R}_-, X)$ into X, and finally, the corresponding *history function* is defined as

$$u_t(s) := u(t+s) \text{ for all } t \geq 0,\ s \leq 0, \text{ with } \varphi \text{ being the initial history data.}$$

It has been known (see, e.g., [**1, 11, 12, 24, 29, 30**]) that, under some certain conditions on B, F and Φ, there exists a corresponding solution semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ on $C_0(\mathbb{R}_-, X)$ such that the solutions to (1.1) have been given by $u_t = T_{B,F,\Phi}(t)\varphi$. If we now consider the function $u : \mathbb{R}_+ \times \mathbb{R}_- \to X$ defined as

$$u(t,s) = [T_{B,F,\Phi}(t)f](s)$$

then we obtain the equality

$$\frac{\partial}{\partial t}u(t,s) = \frac{\partial}{\partial s}u(t,s)$$

which is known as the balance law between the velocity of the evolution process in the past and in the future (see [**4**, p. 39-40]). However, in some applications, e.g., in the biological model on genetic repression proposed by Nobel Prize laureates Jacob and Monod [**18**] (see also Goodwin [**8, 9**]), this balance law may not be true. An idea introduced by Brendle and Nagel [**2**] to control the unbalance is to suppose that the value of the history function is modified according to an evolution law (see [**7**] for the formulation in L_p-spaces). Consequently, this modification leads to the following system of linear partial neutral functional differential equations with nonautonomous past

$$\frac{\partial}{\partial t}F(u(t,\cdot)) = BFu(t,\cdot) + \Phi u(t,\cdot), \quad t \geq 0, \tag{1.2}$$

$$\frac{\partial}{\partial t}(u(t,s)) = \frac{\partial}{\partial s}(u(t,s)) + A(s)u(t,s), \quad t \geq 0 \geq s. \tag{1.3}$$

Here, function $u(\cdot,\cdot)$ takes values in a Banach space X and B is some linear partial differential operator, while the difference operator F and the delay operator Φ are bounded linear operators from the space $C_0(\mathbb{R}_-, X)$ into X, and finally, $A(s)$ are (unbounded) operators on X for which the non-autonomous backward Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} & = -A(t)x(t),\ t \leq s \leq 0, \\ x(s) & = x_s \in X, \end{cases} \tag{1.4}$$

is well-posed with exponential bound. In particular, there exists an exponentially bounded backward evolution family $\mathcal{U} = (U(t,s))_{t\leq s\leq 0}$ solving (1.4), i.e., the solutions of (1.4) are given by $x(t) = U(t,s)x(s)$ for $t \leq s \leq 0$.

In [**14**, Chapt. 4], under these assumptions we have solved the system of Eq. (1.2) and (1.3) by constructing an appropriate strongly continuous evolution semigroup on the space $E := C_0(\mathbb{R}_-, X)$. This semigroup was obtained by proving that a certain operator (see Definition 2.8) satisfies the Hille-Yosida conditions as long as we can write the difference operator as $F = \delta_0 - \Psi$ with Ψ being "small" (see (2.10)). We refer the reader to [**2, 6, 7, 15**] for the well-posedness of delay equations with non-autonomous past, i.e., for the case $\Psi = 0$.

In the present paper, under the above condition on the difference operator F and the smallness of the delay operator Φ, we prove that the solution semigroup for this equation is hyperbolic (or admits an exponential dichotomy) provided that the backward evolution family $\mathcal{U} = (U(t,s))_{t\le s\le 0}$ generated by $-A(s)$ is exponentially stable and the operator B generates a hyperbolic semigroup on X. Furthermore, under the positivity conditions on $(e^{tB})_{t\ge 0}, \Phi, F$ and $U(t,s),\ t \le s \le 0$, we prove that the the above-mentioned solution semigroup is positive. This fact allows to apply the spectral theory of positive semigroups to obtain a spectral criteria for exponential stability of the solution semigroup under consideration. Our results are contained in Theorems 3.6, 4.1 and Corollary 4.8 which extend the results known for delay and neutral functional differential equations (see [**2, 6, 15, 16, 13, 29**]).

2. EVOLUTION SEMIGROUPS WITH DIFFERENCE AND DELAY

In this section, we briefly recall the construction and results obtained in [**14**, Chapt. 4] on the well-posedness of the system of Eq. (1.2) and (1.3) as well as the representation of the resolvent of the evolution semigroup solving this system. We start from an evolution family $\mathcal{U}$ on $\mathbb{R}_-$ and extend it to all of $\mathbb{R}$ in order to define a corresponding evolution semigroup on $C_0(\mathbb{R}, X)$. For most of the concepts of evolution semigroups we refer to the monographs [**3**] or [**5**, Chap. VI.9].

Definition 2.1. *A family of bounded linear operators* $\mathcal{U} = (U(t,s))_{t\le s\le 0}$ *on a Banach space* X *is called a (strongly continuous, exponentially bounded) backward evolution family on* $\mathbb{R}_-$ *if*

(i) $U(t,t) = Id$ *and* $U(t,r)U(r,s) = U(t,s)$ *for* $t \le r \le s \le 0$.

(ii) *the map* $(t,s) \mapsto U(t,s)x$ *is continuous for every* $x \in X$ *with* $(t,s) \in \Delta := \{(t,s) \in \mathbb{R}^2 : t \le s \le 0\}$.

(iii) *there are constants* $H \ge 1$ *and* $\omega_1 \in \mathbb{R}$ *such that*

$$||U(t,s)|| \le He^{\omega_1(s-t)} \quad \text{for all} \quad t \le s \le 0.$$

The constant

$$\omega(\mathcal{U}) := \inf\{\alpha \in \mathbb{R} : \exists H \ge 1 \quad \text{such that} \quad ||U(t,s)|| \le He^{\alpha(s-t)} \quad \forall t \le s \le 0\}$$

is called the growth bound of $\mathcal{U}$. *In case* $\omega(\mathcal{U}) < 0$, *we say that the evolution family* $\mathcal{U}$ *is uniformly exponentially stable.*

This notion of backward evolution families arises when we consider well-posed evolution equations on the negative half-line $\mathbb{R}_-$ of the form

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} &= -A(t)u(t),\ t \le s \le 0, \\ u(s) &= u_s \in X. \end{cases} \tag{2.5}$$

More precisely, we will say that the backward Cauchy problem (2.5) is well-posed with exponential bound if there exists an exponentially bounded backward evolution family $\mathcal{U} = (U(t,s))_{t\le s\le 0}$ solving (2.5), i.e., the solutions of (2.5) are given by $x(t) = U(t,s)x(s)$ for $t \le s \le 0$. Clearly, for backward evolution families on $\mathbb{R}_-$, we have the similar results as in the case of "forward" evolution families on $\mathbb{R}_+$. We refer to [**2, 6, 7, 27**] for detailed treatments of the well-posedness of the evolution equation (2.5). In other words, we may say that the operators $-A(t)$ generate the backward evolution family $\mathcal{U}$.

For later use, we summarize the construction of the corresponding left translation evolution semigroups and some auxiliary results. Firstly, the evolution family $(U(t,s))_{t\le s\le 0}$ is extended to a backward evolution family on $\mathbb{R}$ by setting

$$\tilde{U}(t,s) := \begin{cases} U(t,s) & \text{for} \quad t \le s \le 0, \\ U(t,0) & \text{for} \quad t \le 0 \le s, \\ U(0,0) = Id & \text{for} \quad 0 \le t \le s. \end{cases}$$

Definition 2.2. *On $\tilde{E} := C_0(\mathbb{R}, X)$, we define the left translation evolution semigroup $(\tilde{T}(t))_{t\ge 0}$ corresponding to $(\tilde{U}(t,s))_{t\le s}$ by*

$$(\tilde{T}(t)\tilde{f})(s) := \tilde{U}(s,s+t)\tilde{f}(s+t) = \begin{cases} U(s,s+t)\tilde{f}(s+t) & for \quad s \le s+t \le 0 \\ U(s,0)\tilde{f}(s+t) & for \quad s \le 0 \le s+t \\ \tilde{f}(s+t) & for \quad 0 \le s \le s+t \end{cases}$$

We also denote its generator by $(\tilde{G}, D(\tilde{G}))$.

It can be seen (see [**15**, Lemma 2.5]) that the operator $(\tilde{G}, D(\tilde{G}))$ is a local operator in the sense that if $\tilde{u} \in D(\tilde{G})$ and $\tilde{u}(s) = 0$ for all $a < s < b$, then $[\tilde{G}\tilde{u}](s) = 0$ for all $a < s < b$. Then, locality of $\tilde{G}$ allows us to define an operator G on $E := C_0(\mathbb{R}_-, X)$ as follows.

Definition 2.3. *Take*

$$D(G) := \{\tilde{f}|_{\mathbb{R}_-} : \tilde{f} \in D(\tilde{G})\}$$

and define

$$[Gf](t) := [\tilde{G}\tilde{f}](t) \text{ for } t \le 0 \text{ and } f = \tilde{f}|_{\mathbb{R}_-}.$$

We now have the following description of G taken from [**15**, Lemma 2.5].

Lemma 2.4. *Let $u, f \in E = C_0(\mathbb{R}_-, X)$, and $\lambda \in \mathbb{C}$. Then $u \in D(G)$ and $(\lambda - G)u = f$ if and only if u and f satisfy*

$$u(t) = e^{\lambda(t-s)}U(t,s)u(s) + \int_t^s e^{\lambda(t-\xi)}U(t,\xi)f(\xi)\mathrm{d}\xi \qquad \text{for } t \le s \le 0. \tag{2.6}$$

We note that such an operator G has been used to study the asymptotic behavior of evolution families on the half-line (see, e.g., [**15, 17, 22**]). The operator G becomes a generator only if we restrict it to a smaller domain, e.g., $D := \{u \in D(G) : [Gu](0) = 0\}$ (see [**17**]). However, for later applications we consider a more general case and make the following assumptions.

Assumption 2.5. *On the Banach spaces X and $E := C_0(\mathbb{R}_-, X)$ we consider the following operators.*

(i) *Let $(B, D(B))$ be the generator of a strongly continuous semigroup $(e^{tB})_{t\ge 0}$ on X satisfying $\|e^{tB}\| \le Me^{\omega_2 t}$ for some constants $M \ge 1$ and $\omega_2 \in \mathbb{R}$.*

(ii) *Let the difference operator $F : E \to X$ and the delay operator $\Phi : E \to X$ be bounded and linear.*

Definition 2.6. *On the space E we define a left translation evolution semigroup $(T_{B,0}(t))_{t\ge 0}$ by*

$$[T_{B,0}(t)f](s) = \begin{cases} U(s, s+t)f(s+t), & s+t \le 0, \\ U(s,0)e^{(t+s)B}f(0), & s+t \ge 0, \end{cases}$$

for all $f \in E$.

One can easily verify that $(T_{B,0}(t))_{t\ge 0}$ is strongly continuous. We denote its generator by $G_{B,0}$. We have the following properties of $G_{B,0}$ and $(T_{B,0}(t))_{t\ge 0}$ taken from [**15**, Prop. 2.8].

Proposition 2.7. *The following assertions hold.*

(i) *The generator of $(T_{B,0}(t))_{t\ge 0}$ is given by*

$$D(G_{B,0}) := \{f \in D(G) : f(0) \in D(B) \text{ and } (G(f))(0) = Bf(0)\},$$
$$G_{B,0}f := Gf \text{ for } f \in D(G_{B,0}).$$

(ii) *The set $\{\lambda \in \rho(B) : Re\lambda > \omega(\mathcal{U})\}$ is contained in $\rho(G_{B,0})$. Moreover, for λ in this set, the resolvent $R(\lambda, G_{B,0})$ is given by*

$$[R(\lambda, G_{B,0})f](t) = e^{\lambda t}U(t,0)R(\lambda,B)f(0) + \int_t^0 e^{\lambda(t-\xi)}U(t,\xi)f(\xi)\mathrm{d}\xi \tag{2.7}$$

for $f \in E$, $t \le 0$.

(iii) *The semigroup* $(T_{B,0}(t))_{t\geq 0}$ *satisfies*

$$\|T_{B,0}(t)\| \leq Ke^{\omega t}, \; t \geq 0, \tag{2.8}$$

with the constants $K = MH$ *and* $\omega := max\{\omega_1, \omega_2\}$ *for the constants* M, H, ω_1, ω_2 *appearing in Definition* 2.1 *and Assumption* 2.5.

We then use the difference and delay operators $F,\ \Phi \in \mathcal{L}(E, X)$ to define the following restriction of the operator G from Definition 2.2.

Definition 2.8. *The operator* $G_{B,F,\Phi}$ *is defined by*

$$\begin{aligned} &G_{B,F,\Phi}f := Gf \quad \text{on the domain} \\ &D(G_{B,F,\Phi}) := \{f \in D(G) : Ff \in D(B) \text{ and } F(Gf) = BFf + \Phi f\}. \end{aligned} \tag{2.9}$$

We next write F in the form

$$F\varphi := \varphi(0) - \Psi\varphi, \qquad \varphi \in E, \tag{2.10}$$

for some bounded linear operator $\Psi : E \to X$. The domain of $G_{B,F,\Phi}$ can then be rewritten as

$$D(G_{B,F,\Phi}) = \{f \in D(G) : f(0) - \Psi f \in D(B)$$

and

$$[Gf](0) = B(f(0) - \Psi f) + \Phi f + \Psi Gf\}.$$

If the operator Ψ is "small", we can prove that the resolvent $R(\lambda, G_{B,F,\Phi})$ satisfies the Hille-Yosida estimates yielding that $G_{B,F,\Phi}$ generates a strongly continuous semigroup. This has been done in [**14**, Chapt. 4], and we recall the result on well-posedness of the system of Eq. (1.2) and (1.3) in the following theorem.

Theorem 2.9. [**14**, Thm 4.2, Corollaries 4.3, 4.6] *Let the operator* Ψ *satisfy the condition* $\|\Psi\| < \frac{1}{H}$ *(with the constant* H *as in Definition* 2.1*), and define the operator* $e_\lambda : X \to E$ *by*

$$[e_\lambda x](t) := e^{\lambda t}U(t, 0)x \;\; \text{for } t \leq 0, \; x \in X \;\; \text{and } Re\lambda > \omega(\mathcal{U}).$$

Then the following assertions hold.

(i) $\lambda \in \rho(G_{B,F,\Phi})$ *for all* $\lambda > \omega_1 + \frac{K\|\Phi\|}{1 - H\|\Psi\|}$ *(with the constants* ω_1 *and* K *as in Proposition* 2.7*). For such* λ *the resolvent of* $G_{B,F,\Phi}$ *satisfies*

$$\begin{aligned} R(\lambda, G_{B,F,\Phi})f = {}& e_\lambda[\Psi R(\lambda, G_{B,F,\Phi}) + R(\lambda, B)(\Phi R(\lambda, G_{B,F,\Phi}) - \Psi)]f \\ &+ R(\lambda, G_{B,0})f \quad \text{for } f \in E. \end{aligned} \tag{2.11}$$

(ii) *The operator* $G_{B,F,\Phi}$ *generates a strongly continuous semigroup* $(T_{B,F,\Phi}(t))_{t\geq 0}$ *on* E.

(iii) *The system of Eq.* (1.2) *and* (1.3) *is well-posed. Precisely, for every* $\varphi \in D(G_{B,F,\Phi})$ *there exists a unique classical solution* $u(t,\cdot,\varphi)$ *of* (1.2) *given by*

$$u(t,\cdot,\varphi) = T_{B,F,\Phi}(t)\varphi$$

which satisfies Eq. (1.3) *in a mild sense, i.e., it satisfies*

$$u(t,s,\varphi) = U(s,\tau)u(t,\tau,\varphi) + \int_s^\tau U(s,\xi)\frac{\partial}{\partial t}u(t,\xi,\varphi)\mathrm{d}\xi \quad \textit{for all } t \geq 0 \geq \tau \geq s$$

known as the variation-of-constant formula for Eq. (1.3).

Moreover, for every sequence $(\varphi_n)_{n\in\mathbb{N}} \subset D(G_{B,F,\Phi})$ *satisfying* $\lim_{n\to\infty} \varphi_n = 0$, *one has*

$$\lim_{n\to\infty} u(t,\cdot,\varphi_n) = 0$$

uniformly in compact intervals.

3. SPECTRA AND HYPERBOLICITY OF SOLUTION SEMIGROUPS

Having established the well-posedness of the equation (1.3), we now consider the hyperbolicity of the solution semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$. To do this, we first compute the spectra of the semigroup $(T_{B,0}(t))_{t\geq 0}$ on $E = C_0(\mathbb{R}_-, X)$ and its generator. This will be used to prove the robustness of the hyperbolicity of the semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ under small perturbations by the delay operator Φ. We first compare $(T_{B,0}(t))_{t\geq 0}$ to its restriction to the subspace $C_{00} := \{f \in E : f(0) = 0\}$.

Lemma 3.1. [**15**, Lemma 4.1] *Let the semigroup* $(T_{B,0}(t))_{t\geq 0}$ *on* E *be defined as in Definition* 2.6 *with the generator* $G_{B,0}$. *Denote by* $(T_0(t))_{t\geq 0}$ *the restriction of* $(T_{B,0}(t))_{t\geq 0}$ *to the subspace* C_{00} *and* G_0 *be its generator. Then, the following assertions hold.*

$$\sigma(T_{B,0}(t)) \subseteq \sigma(T_0(t)) \cup \sigma(e^{tB}), \; for \;\; t \geq 0. \tag{3.12}$$

$$\sigma(G_{B,0}) \cup \sigma(B) = \sigma(G_0) \cup \sigma(B) \tag{3.13}$$

In [**22**, Corollary 2.4] it has been proved that a Spectral Mapping Theorem holds for the semigroup $(T_0(t))_{t\geq 0}$. More precisely, we have

$$\sigma(G_0) = \{\lambda \in \mathbb{C} : Re\lambda \leq \omega(\mathcal{U})\}$$

and

$$\sigma(T_0(t))\backslash\{0\} = e^{t\sigma(G_0)}, \text{ for all } t > 0. \tag{3.14}$$

By this and Lemma 3.1 we obtain the following.

Theorem 3.2. [**15**, Theorem 4.2] *Let the operators G_0 be defined as in Lemma 3.1. Then the spectral equality*

$$[\sigma(T_{B,0})(t) \cup \sigma(e^{tB})]\backslash\{0\} = [e^{t\sigma(G_0)} \cup \sigma(e^{tB})]\backslash\{0\}, \quad t \geq 0 \tag{3.15}$$

holds.

Therefore, using the spectral characterization of hyperbolic semigroups (see [**5**, Theorem V.1.15]), the above theorem allows the following consequence.

Corollary 3.3. *If the operator $(B,D(B))$ generates a hyperbolic semigroup $(e^{tB})_{t\geq 0}$ and if the backward evolution family $\mathcal{U} = (U(t,s))_{t\leq s\leq 0}$ is uniformly exponentially stable, then the semigroup $(T_{B,0}(t))_{t\geq 0}$ is hyperbolic.*

Proof. The assumption that $\mathcal{U}$ is uniformly exponentially stable means that $\omega(\mathcal{U}) < 0$, hence $s(G_0) < 0$ by (3.14). Therefore, $\sigma(G_0) \cap iR = \emptyset$. By the hyperbolicity of $(e^{tB})_{t\geq 0}$ we have

$$(e^{t\sigma(G_0)} \cup \sigma(e^{tB})) \cap e^{i\mathbb{R}} = \emptyset.$$

The hyperbolicity of $(T_{B,0}(t))_{t\geq 0}$ follows from (3.15) and [**5**, Theorem V.1.15]. □

The main purpose of this section is to prove the existence of hyperbolicity of the solution semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ under the conditions that the semigroup $(e^{tB})_{t\geq 0}$ is hyperbolic and the delay operator Φ has sufficiently small norm. To do so, we need the following characterization of hyperbolic semigroups (see [**25**, Theorem 2.6.2]).

Theorem 3.4. *Let the $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with the generator A. Then the following assertions are equivalent.*

(i) $(T(t))_{t\geq 0}$ *is hyperbolic.*

(ii) $i\mathbb{R} \subset \rho(A)$ *and*

$$(C,1)\sum_{k\in\mathbb{Z}} R(i\omega + ik, A)x := \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(i\omega + ik, A)x$$

converges for all $\omega \in \mathbb{R}$ and $x \in X$.

We note that the above theorem is taken from [**25**, Theorem 2.6.2], while its proof is essentially due to G. Greiner and M. Schwarz [**10**, Theorem 1.1 and Corollary 1.2]. A continuous version of the above theorem is proved by M. Kaashoek and S. Verduyn Lunel in [**19**, Theorem 4.1] . In order to apply this theorem we have to compute the resolvent $R(\lambda, G_{B,F,\Phi})$ starting from the resolvents $R(\lambda, G_{B,0})$ and $R(\lambda, G_B)$. This can be done as follows.

Lemma 3.5. *Let the backward evolution family $\mathcal{U}$ be uniformly exponentially stable and the operator $(B, D(B))$ generate a hyperbolic semigroup $(e^{tB})_{t\geq 0}$. Then, if $\|\Psi\| < 1/K_1$ with K_1 being given in (3.17) below, and $||\Phi||$ is sufficiently small, then there exists an open strip Σ containing the imaginary axis and a function H_λ which is analytic and uniformly bounded on Σ such that*

$$R(\lambda, G_{B,F,\Phi}) = H_\lambda[R(\lambda, G_{B,0}) - e_\lambda R(\lambda, B)\Psi] \text{ for } \lambda \in \Sigma. \tag{3.16}$$

Proof. By [**19**, Theorem 4.1] and the hyperbolicity of $(e^{tB})_{t\geq 0}$ we obtain that, there exist constants $P_1,\ \nu$ such that

$$||R(\lambda, B)|| \leq P_1 \text{ for all } \quad |Re\lambda| < \nu.$$

By the uniformly exponential stability of $\mathcal{U}$, there exist constants $\omega_1 > 0$ and K_1 such that

$$\|U(t, s)\| < K_1 e^{-\omega_1(s-t)} \quad \text{for all} \quad t \leq s \leq 0. \tag{3.17}$$

Let now ω be a real number such that $0 < \omega < min\{\omega_1, \nu\}$. We then put

$$\Sigma := \{\lambda \in \mathbb{C} : |Re\lambda| < \nu\}$$

and

$$P := \sup_{\lambda\in\Sigma} ||R(\lambda, B)||. \tag{3.18}$$

We first verify that for each $f \in E$ and $\lambda \in \Sigma$ the equation $u = e_\lambda\big[\Psi u + R(\lambda, B)\Phi u\big] - e_\lambda R(\lambda, B)\Psi f + R(\lambda, G_{B,0})f$ has a unique solution $u \in E$. Indeed, let $M_\lambda : E \to E$ be the linear operator defined as $M_\lambda := e_\lambda(\Psi + R(\lambda, B)\Phi)$ with e_λ as in Theorem 2.9. For $\lambda \in \Sigma$ this operator is bounded and satisfies

$$||M_\lambda|| \leq K_1(||\Psi|| + P||\Phi||) < 1 \text{ if, in addition, } ||\Phi|| < \frac{1 - K_1||\Psi||}{PK_1}.$$

Therefore, the operator $I - M_\lambda$ is invertible, and the equation

$$u = e_\lambda\big[\Psi u + R(\lambda, B)\Phi u\big] - e_\lambda R(\lambda, B)\Psi f + R(\lambda, G_{B,0})f$$

has a unique solution

$$u = (I - M_\lambda)^{-1}[R(\lambda, G_{B,0})f - e_\lambda R(\lambda, B)\Psi f].$$

Putting $H_\lambda := (I - M_\lambda)^{-1}$ we obtain

$$R(\lambda, G_{B,F,\Phi}) = H_\lambda[R(\lambda, G_{B,0}) - e_\lambda R(\lambda, B)\Psi].$$

Since

$$H_\lambda = (I - M_\lambda)^{-1} = \sum_{n=0}^{\infty} M_\lambda^n \tag{3.19}$$

it follows that

$$||H_\lambda|| \le \sum_{n=0}^{\infty} ||M_\lambda||^n \le \sum_{n=0}^{\infty}[K_1(||\Psi|| + P||\Phi||)]^n$$
$$= \frac{1}{1 - K_1(||\Psi|| + P||\Phi||)} \qquad \text{for all } \lambda \in \Sigma, \text{ as } ||\Phi|| < \frac{1 - K_1||\Psi||}{PK_1}.$$

Since $||M_\lambda^n|| \le [K_1(||\Psi|| + P||\Phi||)]^n$ for all $\lambda \in \Sigma$ and the series

$$\sum_{n=0}^{\infty}[K_1(||\Psi|| + P||\Phi||)]^n$$

converges for $||\Phi|| < \frac{1 - K_1||\Psi||}{PK_1}$ we obtain that, if $||\Phi|| < \frac{1 - K_1||\Psi||}{PK_1}$, then the Neumann series (3.19) converges uniformly for all $\lambda \in \Sigma$. This fact, together with the analyticity of M_λ, yields the analyticity of H_λ. □

Using the relation (3.16) and representations of resolvents we obtain the following result on the hyperbolicity of the solution semigroup $(T_{B,F,\Phi}(t))_{t\ge 0}$.

Theorem 3.6. *Let the assumptions of Theorem* 2.9 *be satisfied. In addition, let the backward evolution family $\mathcal{U}$ be uniformly exponentially stable and the operator $(B, D(B))$ be the generator of a hyperbolic C_0-semigroup $(e^{tB})_{t\ge 0}$, and the norm of the operator Ψ satisfy $\|\Psi\| < \frac{1}{K_1}$. Then, if the norm of the delay operator Φ is sufficiently small, the solution semigroup $(T_{B,F,\Phi}(t))_{t\ge 0}$ is hyperbolic.*

Proof. By Corollary 3.3, the semigroup $(T_{B,0}(t))_{t\ge 0}$ is hyperbolic. We first prove that, for sufficiently small $||\Phi||$ the sum $\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}[R(i\omega + ik, G_{B,F,\Phi})]$ is bounded in $\mathcal{L}(E)$. In fact, by Lemma 3.5, we have

$$\text{(3.20)} \quad \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}[R(i\omega + ik, G_{B,F,\Phi})f](s)$$
$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\big\{(1 + M_{i\omega+ik} + M_{i\omega+ik}^2 + \cdots)[R(i\omega + ik, G_{B,0})$$
$$- e_{i\omega+ik}R(i\omega + ik, B)\Psi]f\big\}(s)$$
$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\big\{[R(i\omega + ik, G_{B,0}) - e_{i\omega+ik}R(i\omega + ik, B)\Psi]f\big\}(s)+$$
$$+ \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{(i\omega+ik)s}U(s,0)(\Psi + R(i\omega + ik, B)\Phi)\{[R(i\omega + ik, G_{B,0})$$
$$- e_{i\omega+ik}R(i\omega + ik, B)\Psi]f\} + \cdots$$

for $s \in \mathbb{R}_-$.

Note that the semigroup $(T_{B,0})_{t\geq 0}$ is hyperbolic, hence $e^{-2\pi i\omega} \in \rho(T_{B,0}(2\pi))$ for all $\omega \in \mathbb{R}$. Using the formula (see [**5**, Lemma II.1.9])

$$R(\lambda, G_{B,0})(1 - e^{-\lambda t}T_{B,0}(t)) = \int_0^t e^{-\lambda s}T_{B,0}(s)\mathrm{d}s, \quad \lambda \in \rho(G_{B,0})$$

we obtain

$$\begin{aligned} R(i\omega + ik, G_{B,0}) &= \int_0^{2\pi} e^{-(i\omega+ik)t}T_{B,0}(t)(1 - e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}\mathrm{d}t, \\ R(i\omega + ik, B) &= \int_0^{2\pi} e^{-(i\omega+ik)t}e^{tB}(1 - e^{2\pi B})^{-1}\mathrm{d}t. \end{aligned}$$

The first term of (3.20) can now be computed as

$$\begin{aligned} &\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}[R(i\omega + ik, G_{B,0}) - e_{i\omega+ik}R(i\omega + ik, B)\Psi]f \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\int_0^{2\pi} e^{-(i\omega+ik)t}[T_{B,0}(t)(1 - e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1} \\ &\quad - e_{(i\omega+ik)}e^{tB}(1 - e^{2\pi B})^{-1}\Psi]f\,\mathrm{d}t \\ &= \int_0^{2\pi}\left[\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{-ikt}\right]e^{-i\omega t}[T_{B,0}(t)(1 - e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1} \\ &\quad - e_{(i\omega+ik)}e^{tB}(1 - e^{2\pi B})^{-1}\Psi]f\,\mathrm{d}t \\ &= \int_0^{2\pi}\sigma_N(t)e^{-i\omega t}[T_{B,0}(t)(1 - e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1} - e_{(i\omega+ik)}e^{tB}(1 - e^{2\pi B})^{-1}\Psi]f\,\mathrm{d}t, \end{aligned}$$

here, $\sigma_N(t) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{-ikt}$. Since

$$\sigma_N(t) = \frac{1 - \cos(Nt)}{N(1 - \cos t)} \geq 0 \text{ and } \int_0^{2\pi}\sigma_N(t)\mathrm{d}t = 2\pi \tag{3.21}$$

(see [**10**, Theorem 1.1]), the norm of the first term in (3.20) can be estimated by

$$\left\|\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\left[R(i\omega + ik, G_{B,0}) - e_{i\omega+ik}R(i\omega + ik, B)\Psi\right]f\right\| \leq C_1||f|| \tag{3.22}$$

with

$$\begin{aligned} C_1 &:= 2\pi \sup_{0\leq\omega\leq 1}\{||(1 - e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}|| + ||(1 - e^{2\pi B})^{-1}||||\Psi||\} \\ &\quad\times \sup_{0\leq t\leq 2\pi}\{||T_{B,0}(t)|| + ||e^{tB}||\} \end{aligned}$$

We now compute the second term of (3.20). For $s \in \mathbb{R}_-$ we have

$$\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} M_{(i\omega+ik)}[(R(i\omega+ik,G_{B,0})-e_{i\omega+ik}R(i\omega+ik,B)\Psi)f](s)\\
&=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{(i\omega+ik)s}U(s,0)(\Psi+R(i\omega+ik,B)\Phi)[R(i\omega+ik,G_{B,0})\\
&-e_{i\omega+ik}R(i\omega+ik,B)\Psi]f\\
&=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{(i\omega+ik)s}U(s,0)\left(\Psi+\int_0^{2\pi} e^{-(i\omega+ik)\tau}e^{\tau B}(1-e^{2\pi B})^{-1}\mathrm{d}\tau\Phi\right)\\
&\times\int_0^{2\pi} e^{-(i\omega+ik)t}\left[T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}-e_{(i\omega+ik)}e^{tB}(1-e^{2\pi B})^{-1}\Psi\right]\mathrm{d}t\\
&=\int_0^{2\pi}\left[\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{-ik(t-s)}\right]e^{-i\omega(t-s)}U(s,0)\Psi\Big[T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}\\
&-e_{(i\omega+ik)t}e^{tB}(1-e^{2\pi B})^{-1}\Psi\Big]\mathrm{d}t+\int_0^{2\pi}\int_0^{2\pi}\left[\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{-ik(t+\tau-s)}\right]\\
&\times e^{-i\omega(t+\tau-s)}U(s,0)e^{\tau B}(1-e^{2\pi B})^{-1}\Phi\\
&\times\left[T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}-e_{(i\omega+ik)t}e^{tB}(1-e^{2\pi B})^{-1}\Psi\right]\mathrm{d}\tau\,\mathrm{d}t\\
&=\int_0^{2\pi}\sigma_N(t-s)e^{-i\omega(t-s)}U(s,0)\Psi\Big[T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}\\
&-e_{(i\omega+ik)t}e^{tB}(1-e^{2\pi B})^{-1}\Psi\Big]dt\\
&+\int_0^{2\pi}\int_0^{2\pi}\sigma_N(t+\tau-s)e^{-i\omega(t+\tau-s)}U(s,0)e^{\tau B}(1-e^{2\pi B})^{-1}\Phi\\
&\times\left[T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}-e_{(i\omega+ik)t}e^{tB}(1-e^{2\pi B})^{-1}\Psi\right]\mathrm{d}\tau\,\mathrm{d}t.
\end{aligned}$$

Therefore, using (3.21), the norm of the second term of (3.20) can be estimated by

$$C_1(||\Psi||+C_2K_1||\Phi||)||f||$$

with $C_2 := 2\pi||(1-e^{2\pi B})^{-1}||\sup_{0\le t\le 2\pi}\{||e^{tB}||\}$ and C_1 as in (3.22). By induction, the norm of the n^{th} term of (3.20) is estimated by

$$C_1(||\Psi||+C_2K_1||\Phi||)^n||f||.$$

Moreover, the series $\sum_{n=0}^{\infty} C_1(||\Psi||+C_2K_1||\Phi||)^n$ converges if $||\Phi|| < \dfrac{1-||\Psi||}{C_2K_1}$.

Hence, for these $||\Phi||$ the sum $\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(i\omega+ik, G_{B,F,\Phi})$ is bounded in $\mathcal{L}(E)$. We now prove the convergence of $(C,1)\sum_{k\in Z} R(i\omega+ik, G_{B,F,\Phi})f$ for $\omega\in\mathbb{R},\ f\in E$. This can be done by using the idea from [**10**, Theorem 1.1]. By [**26**, III.4.5], it is sufficient to show convergence on a dense subset. From the fact that $i\mathbb{R}\subset\rho(G_{B,F,\Phi})$ and by the spectral mapping theorem for the residual spectrum (see [**5**, Theorem IV.3.7]) we obtain that $e^{-2\pi i\omega}$ does not belong to the residual spectrum $R\sigma(T_{B,F,\Phi})$. This implies that $(1-e^{-2\pi i\omega}T_{B,F,\Phi}(2\pi))E$ is a dense subset of E. Let $f=(1-e^{-2\pi i\omega}T_{B,F,\Phi}(2\pi))g$. Then

$$\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(i\omega+ik, G_{B,F,\Phi})(1-e^{-2\pi i\omega}T_{B,F,\Phi}(2\pi))g) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\int_0^{2\pi} e^{-(i\omega+ik)s}T_{B,F,\Phi}(s)g\,\mathrm{d}s. \tag{3.23}$$

Now $e^{-i\omega}T_{B,F,\Phi}(.)g$ is a continuous function with Fourier coefficients

$$Q_k=\frac{1}{2\pi}\int_0^{2\pi} e^{-(i\omega+ik)s}T_{B,F,\Phi}(s)g\,\mathrm{d}s.$$

Therefore, by Fejer's Theorem [**21**, Theorem I.3.1] , the sum in (3.23) converges as $N\to\infty$.

The assertion of the theorem now follows from Theorem 3.4. □

We illustrate our result by the following example.

Example 3.7. Consider the neutral partial differential equation

$$\begin{aligned}
&\frac{\partial w(x,t,0)}{\partial t}-k\frac{\partial w(x,t,-1)}{\partial t}=\frac{\partial^2 w(x,t,0)}{\partial x^2}-k\frac{\partial^2 w(x,t,-1)}{\partial x^2}+\alpha(w(x,t,0)-kw(x,t,-1))\\
&\qquad\qquad+\int_{-\infty}^{0}\psi(s)w(x,t,s)\mathrm{d}s \quad\text{for } 0\le x\le\pi,\quad t\ge 0,\\
&w(0,t,s)=w(\pi,t,s)=0,\qquad t\ge 0\ge s,\\
&\frac{\partial w(x,t,s)}{\partial t}=\frac{\partial w(x,t,s)}{\partial s}-a(s)\frac{\partial^2 w(x,t,s)}{\partial x^2}\ \text{for all } x\in[0,\pi],\ t\ge 0\ge s
\end{aligned}\tag{3.24}$$

where k and α are real constants with $|k|<1$, $\alpha>1$ and $\alpha\neq n^2$ for all $n\in\mathbb{N}$; the functions ψ and φ are given such that $\psi\in L_1(\mathbb{R}_-)$ and φ is continuous; lastly, the function $a(\cdot)\in L_{1,loc}(\mathbb{R}_-)$ satisfies $a(\cdot)\ge\gamma>0$ for some constant γ.

We choose the Hilbert space $X:=L_2[0,\pi]$ and let $B:D(B)\subset X\to X$ be defined by $B(f)=f''+\alpha f$ with the domain

$$D(B)=H_0^2[0,\pi]:=\{f\in W^{2,2}[0,\pi]: f(0)=f(\pi)=0\}.$$

Also define the difference and delay operators F and Φ as

$$F : C_0(\mathbb{R}_-, X) \to X, \ F(f) := f(0) - kf(-1)$$
$$\Phi : C_0(\mathbb{R}_-, X) \to X, \ \Phi(f) := \int_{-\infty}^{0} \psi(s)f(s)\mathrm{d}s.$$

Clearly, F and Φ are bounded linear operators. Moreover, $\|\Phi\| \leq \|\psi\|_{L_1}$.

We then take operators $A(s) := -a(s)\Delta, \ s \leq 0$, where $\Delta(f) = f''$ with the domain $D(\Delta) = H_0^2[0,\pi]$. The operators $-A(s)$ generate a backward evolution family $(U(r,s))_{r\leq s\leq 0}$ given by

$$U(r,s) = e^{(\int_r^s a(\tau)\mathrm{d}\tau)\Delta} \text{ for all } r \leq s \leq 0.$$

We then have

$$\|U(r,s)\| \leq e^{-\int_r^s a(\tau)\mathrm{d}\tau} \leq e^{-\gamma(s-r)} \text{ for all } r \leq s \leq 0.$$

Hence, we can choose in Definition 2.1 the constants $H = 1$ and $\omega_1 = -\gamma < 0$. The backward evolution family $(U(r,s))_{r\leq s\leq 0}$ is therefore uniformly exponentially stable.

The system (3.24) can now be rewritten as

$$\frac{\partial}{\partial t}F(u(t,\cdot)) = BFu(t,\cdot) + \Phi u(t,\cdot), \quad t \geq 0, \tag{3.25}$$
$$\frac{\partial}{\partial t}(u(t,s)) = \frac{\partial}{\partial s}(u(t,s)) + A(s)u(t,s), \quad t \geq 0 \geq s \tag{3.26}$$

for $u(t,s) = w(\cdot,t,s)$.

It can be seen (see [**5**]) that B is the generator of an analytic semigroup $(e^{tB})_{t\geq 0}$.

Since $\sigma(B) = \{-1+\alpha, -4+\alpha, \ldots, -n^2+\alpha, \ldots\}$ and $\alpha \neq n^2$ for all $n \in \mathbb{N}$, it follows that $\sigma(B) \cap i\mathbb{R} = \emptyset$. Therefore, applying the spectral mapping theorem for analytic semigroups we obtain that the semigroup $(e^{tB})_{t\geq 0}$ is hyperbolic.

Theorem 3.6 now yields that, if $\|\psi\|_{L_1} < \dfrac{1-|k|}{2\pi||(1-e^{2\pi B})^{-1}|| \sup\limits_{0\leq t\leq 2\pi}\{||e^{tB}||\}}$, then the solution semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ for the system of Eq. (3.25) and (3.26) is also hyperbolic.

4. POSITIVITY OF SOLUTION SEMIGROUPS

In this section, we assume X to be a Banach lattice. Then E becomes a Banach lattice as well. Furthermore, we suppose that the semigroup $(e^{tB})_{t\geq 0}$ generated by B, the delay operator Φ and the difference operator F are all positive. Finally, we assume that the backward evolution family $(U(t,s))_{t\leq s\leq 0}$ consists of positive operators. For general theory of positive semigroups we refer to [**23**], [**5**, Chap. VI.1.b], and [**28**]. We then arrive at the following result on the positivity of the solution semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$.

Theorem 4.1. *Let B generate a positive semigroup $(e^{tB})_{t\geq 0}$ on X. Suppose that the operators Φ, Ψ, F and $U(t,s)$, $t \leq s \leq 0$, are all positive with the norm $\|\Psi\| < 1/H$. Then, the semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ generated by $G_{B,F,\Phi}$ is also positive.*

Proof. By Hille-Yosida Theorem for the generator B we have that $\|R(\lambda, B)\| \leq \dfrac{M}{\lambda - w_2}$ for all $\lambda > w_2$. Let $M_\lambda := e_\lambda(\Psi + R(\lambda, B)\Phi)$ with e_λ as in Theorem 2.9. Since $U(t,0)$, Ψ, Φ and $R(\lambda, B)$ are all positive, we obtain that M_λ is also positive. Moreover, for $\lambda > w_2 + \dfrac{MH\|\Phi\|}{1 - H\|\Psi\|}$ we have

$$||M_\lambda|| \leq K_1\bigg(||\Psi|| + \frac{M}{\lambda - w_2}||\Phi||\bigg) < 1.$$

Therefore, the operator $I - M_\lambda$ is invertible for $\lambda > w_2 + \dfrac{MH\|\Phi\|}{1 - H\|\Psi\|}$, and using the Neumann's series we have that the inverse

$$(I - M_\lambda)^{-1} = \sum_{n=0}^{\infty} M_\lambda^n \tag{4.27}$$

is also positive. From (2.11) we arrive at

$$R(\lambda, G_{B,F,\Phi}) = (I - M_\lambda)^{-1}[R(\lambda, G_{B,0}) - e_\lambda R(\lambda, B)\Psi].$$

Using the formula (2.7) for $R(\lambda, G_{B,0})$ we obtain that

$$R(\lambda, G_{B,F,\Phi}) = (I - M_\lambda)^{-1}[e_\lambda R(\lambda, B)F + V_\lambda]$$

where V_λ is defined by

$$[V_\lambda f](t) := \int_t^0 e^{\lambda(t-\xi)} U(t,\xi) f(\xi)\, \mathrm{d}\xi \quad \text{for } f \in E,\ t \leq 0. \tag{4.28}$$

Obviously, V_λ belongs to $\mathcal{L}(E, X)$ and is positive. Therefore, $R(\lambda, G_{B,F,\Phi})$ is positive for λ large enough. Thus, by [**5**, Charac. Thm. VI.1.8] we have that $G_{B,F,\Phi}$ generates a positive semigroup. ☐

We then study the relation between the spectra $\sigma(G_{B,F,\Phi})$ and $\sigma(BFe_\lambda + \lambda\Psi e_\lambda + \Phi e_\lambda)$.

Proposition 4.2. *Let Ψ take the values in $D(B)$. Then, for every complex number λ we have*

$$\lambda \in \sigma(G_{B,F,\Phi}) \quad \textit{if and only if} \quad \lambda \in \sigma(BFe_\lambda + \lambda\Psi e_\lambda + \Phi e_\lambda).$$

Proof. From the definition of a resolvent set we have that $\lambda \in \rho(G_{B,F,\Phi})$ if and only if for every $f \in E$, there exists a unique solution $u \in D(G_{B,F,\Phi})$ of the equation

$$u(t) = e^{\lambda(t-s)} U(t,s) u(s) + \int_t^s e^{\lambda(t-\xi)} U(t,\xi) f(\xi)\, \mathrm{d}\xi \ \text{ for } t \leq s \leq 0.$$

We can obviously rewrite the above equation in an equivalent form as

$$u = e_\lambda y + V_\lambda f \tag{4.29}$$

for some $y = u(0) \in X$, where V_λ is defined as in (4.28). On the other hand, since Ψ takes values in $D(B)$, $u \in D(G_{B,\Phi,F})$ if and only if $y \in D(B)$ and

$$[Gu](0) = B(y - \Psi u) + \Phi u + \Psi Gu.$$

Using the fact $Gu = \lambda u - f$ we obtain

$$\lambda y - f(0) = B(y - \Psi e_\lambda y - \Psi V_\lambda f) + \Phi e_\lambda y + \Phi V_\lambda f + \lambda(\Psi e_\lambda y + \Psi V_\lambda f) - \Psi f.$$

This shows that $\lambda \in \rho(G_{B,F,\Phi})$ if and only if for every $f \in E$ there exists a unique $y \in X$ such that

$$(\lambda - BFe_\lambda - \Phi e_\lambda - \lambda \Psi e_\lambda)y = S_\lambda f,$$

where $S_\lambda := \delta_0 + B\Psi V_\lambda + \lambda \Psi V_\lambda - \Psi + \Phi V_\lambda \in \mathcal{L}(E, X)$. Hence, the proof is complete if we can show that S_λ is surjective from E to X. This actually follows from

$$S_\lambda e_\mu = I + (B\Psi V_\lambda + \lambda \Psi V_\lambda - \Psi + \Phi V_\lambda)e_\mu \in \mathcal{L}(X),$$

since $\|(B\Psi V_\lambda + \lambda \Psi V_\lambda - \Psi + \Phi V_\lambda)e_\mu\| \to 0$ as $\mu \to \infty$, and hence $S_\lambda e_\mu$ is bijective for μ sufficiently large. □

We next recall some notions from stability theory of positive semigroups. Firstly, we have the concept of the spectral bound of a closed operator as follows.

Definition 4.3. *Let $A : D(A) \subset X \to X$ be a closed operator on a Banach space X. Then*

$$s(A) := \sup\{Re\lambda : \lambda \in \sigma(A)\}$$

is called the spectral bound of A.

We then have the following notions of the exponential stability of a semigroup.

Definition 4.4. *A strongly continuous semigroup $(T(t))_{t\geq 0}$ with the generator $(A, D(A))$ is called exponentially stable if there exists an $\epsilon > 0$ such that*

$$\lim_{t\to\infty} \|e^{\epsilon t} T(t)x\| = 0 \text{ for all } x \in D(A).$$

We note that, generally, the condition $s(A) < 0$ doesn't imply the exponential stability of a strongly continuous semigroup $(T(t))_{t\geq 0}$ generated by A (see, e.g., [**25**, Example 1.2.4]). However, for positive strongly continuous semigroups this implication holds true. More precisely, we have the following theorem taken from [**5**, Chapt. VI, Proposition 1.14].

Theorem 4.5. [**5**, Chapt. VI, Proposition 1.14] *Let $(T(t))_{t\geq 0}$ be a positive strongly continuous semigroup with the generator $(A, D(A))$ on a Banach lattice X. Then the spectral bound $s(A)$ satisfies $s(A) < 0$ if and only if $(T(t))_{t\geq 0}$ is exponentially stable in the sense of Definition 4.4.*

We next describe the behavior of the *spectral function* $s : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ defined by $s(\lambda) := s(BFe_\lambda + \lambda\Psi e_\lambda + \Phi e_\lambda)$ for $\lambda \in \mathbb{R}$ in the following proposition whose proof can be done by the same way as in [**5**, Proposition VI.6.13].

Proposition 4.6. *Under the hypotheses of Theorem* 4.1 *and Proposition* 4.2, *if the operator-valued function* $S(\lambda) = \lambda\Psi e_\lambda$ *is decreasing for* $\lambda \in \mathbb{R}$, *then the spectral bound function* $s(\cdot)$ *is decreasing and continuous from the left on* $\mathbb{R}$.

We now come to our next results on the relation between the spectral bounds $s(G_{B,F,\Phi})$ and $s(BFe_\lambda + \lambda\Psi e_\lambda + \Phi e_\lambda)$.

Theorem 4.7. *Under the hypotheses of Proposition* 4.6, *for* $\lambda \in \mathbb{R}$ *we have that, if* $s(BFe_\lambda + \lambda\Psi e_\lambda + \Phi e_\lambda) < \lambda$, *then* $s(G_{B,F,\Phi}) < \lambda$.

Proof. Let $\lambda > s(BFe_\lambda + \lambda\Psi e_\lambda + \Phi e_\lambda)$. Then we obtain from monotonicity of $s(\cdot)$ (see Proposition 4.6) that

$$\mu \geq \lambda > s(BFe_\lambda + \lambda\Psi e_\lambda + \Phi e_\lambda) \geq s(BFe_\mu + \mu\Psi e_\mu + \Phi e_\mu)$$

for all $\mu \geq \lambda$. This yields that $\mu \in \rho(BFe_\mu + \mu\Psi e_\mu + \Phi e_\mu)$ and therefore $\mu \in \rho(A)$ for all $\mu \geq \lambda$ by Theorem 4.1. On the other hand, since $G_{B,F,\Phi}$ generates a positive semigroup, it follows from [**5**, Theorem VI.1.10] that $s(G_{B,F,\Phi}) \in \sigma(G_{B,F,\Phi})$, hence $\lambda > s(G_{B,F,\Phi})$. ☐

Theorems 4.5 and 4.7 now yield the following corollary on a sufficient condition for the exponential stability (in the sense of Definition 4.4) of the semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ generated by $G_{B,F,\Phi}$.

Corollary 4.8. *Let the hypotheses of Proposition* 4.6 *be satisfied. Then the semigroup* $(T_{B,F,\Phi}(t))_{t\geq 0}$ *is exponentially stable in the sense of Definition* 4.4 *if the spectral bound* $s(BFe_0 + \Phi e_0)$ *is less than* 0.

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