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LAPLACIAN COEFFICIENTS OF UNICYCLIC GRAPHS WITH THE NUMBER OF LEAVES AND GIRTH

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Motivated by Ilić and Ilić's conjecture [A. Ilić, M. Ilić: *Laplacian coefficients of trees with given number of leaves or vertices of degree two.* Linear Algebra Appl., 431 (2009), 2195–2202.], we investigate properties of the minimal elements in the partial set $(\mathcal{U}_{n,\ell}^g, \preceq)$ of the Laplacian coefficients, where $\mathcal{U}_{n,\ell}^g$ denote the set of n-vertex unicyclic graphs with the number of leaves ℓ and girth g. These results are used to disprove their conjecture. Moreover, the graphs with minimum Laplacian-like energy in $\mathcal{U}_{n,\ell}^g$ are also studied.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices and m edges and $L(G) = D(G) - A(G)$ be its *Laplacian matrix,* where $A(G)$ and $D(G)$ are its adjacency and degree diagonal matrices, respectively. The *Laplacian polynomial* $\mathcal{L}(G, \lambda)$ of G is the characteristic polynomial of its Laplacian matrix $L(G)$, i.e.,

$$\mathcal{L}(G,\lambda) = \det(\lambda I_n - L(G)) = \sum_{k=0}^{n} (-1)^k c_k(G) \lambda^{n-k}.$$

Then $L(G)$ has nonnegative eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. From Viette's formula, $c_k = \sigma_k(\mu_1, \mu_2 \cdots \mu_{n-1})$ is a symmetric polynomial of order $n-1$. In particular, we have $c_0 = 1, c_n = 0, c_1 = 2|E(G)|, c_{n-1} = n\tau(G)$, where $\tau(G)$ is the number of spanning trees of G (see [**10**]). If G is a tree, the Laplacian

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coefficient c_{n-2} is equal to its Wiener index, which is the sum of all distances between unordered pairs of vertices of G (see [**1**], [**16**]).

$$c_{n-2}(T) = W(T) = \sum_{u,v \in V} d(u,v).$$

In general, Laplacian coefficients c_k can be expressed in terms of subtree structures of G.

Theorem 1.1. [**8**] *Let $\mathcal{F}_k$ be the set of all spanning forests of G with exactly k components. The Laplacian coefficient c_{n-k} of a graph G is expressed by $c_{n-k}(G) = \sum_{F \in \mathcal{F}_k} \gamma(F)$, where F has k components T_i with n_i vertices, $i = 1, 2, \ldots, k$ and $\gamma(F) = \prod_{i=1}^{k} n_i$.*

Recently, the study on the Laplacian coefficients has attracted much attention. Let G and H be two graphs of order n. We write $G \preceq H$ if $c_k(G) \leq c_k(H)$ for all $0 \leq k \leq n$, and write $G \prec H$ if $G \preceq H$ and $c_k(G) < c_k(H)$ for some $k \in \{0, 1, \ldots, n\}$. Mohar [**11**] first investigated properties of the poset (partially ordered set) of acyclic graphs with the partial order $\preceq$ and proposed several questions. Later, Ilić [**4**] and Zhang et al. [**17**] investigated ordering trees by the Laplacian coefficients. Stevanović and Ilić [**13**] investigated and characterized the minimum and maximum elements in the poset of unicyclic graphs of order n with $\preceq$. Tan [**15**] proved that the poset of unicyclic graphs of order n and fixed matching number with $\preceq$ has only one minimal element. He and Shan [**3**] studied the properties of the poset of bicyclic graphs of order n.

The Laplacian-like energy in [**9**] of G, LEL for short, is defined as follows:

$$LEL(G) = \sum_{k=1}^{n-1} \sqrt{\mu_k},$$

since it has similar features as molecular graph energy defined by Gutman [**2**]. LEL describes well the properties which have a close relation with the molecular structures and was proved to be as good as the Randić index, better than Wiener index in some areas (see [**14**]). Further, Stevanović [**12**, Lemma 2] established a connection between LEL and Laplacian coefficients. Later, Ilić, Krtinić and Ilić [**7**, Theorem 1.1] provided a corrected proof of the following fact:

Theorem 1.2. [**12, 7**] *Let G and H be two graphs with n vertices. If $G \preceq H$, then $LEL(G) \leq LEL(H)$. Furthermore, if $G \prec H$, then $LEL(G) < LEL(H)$.*

Denote by

$\mathcal{U}_{n,\ell} = \{G \mid G$ is a n-vertex unicyclic graph with fixed ℓ leaves $\}$,
$\mathcal{U}_{n,\ell}^{g} = \{G \mid G \in \mathcal{U}_{n,\ell}$ with fixed girth g $\}$.

Let $BST_{n,\ell}$ be a balanced starlike tree of order n with ℓ leaves which is obtained by identifying one end of each of the ℓ paths of orders $\left\lfloor \frac{n-1}{\ell} \right\rfloor + 1$ or $\left\lceil \frac{n-1}{\ell} \right\rceil + 1$. Moreover, let $U_{n,\ell}^{g,p}$ (see Fig. 1) be a balanced starlike unicyclic graph of order n with ℓ leaves and girth g, which is obtained by identifying one end of a path P_{p+1} of order $p+1$ and one vertex of a cycle of order g, the other end of P_{p+1} and the center vertex of a balanced starlike tree $BST_{n-p-g+1,\ell}$, respectively.

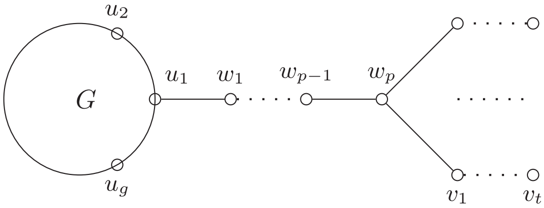
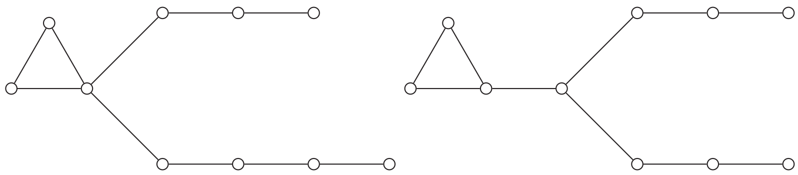


Figure 1. Graph $U_{n,\ell}^{g,p}$

Ilić and Ilić [**6**] proposed the following conjecture:

Conjecture 1.3. [**6**] *Among all n-vertex unicyclic graphs, the graph* $U_{n,\ell}^{3,0}$ *has the minimum Laplacian coefficients* c_k, $k = 0, \ldots, n$, *i.e.,* $U_{n,\ell}^{3,0}$ *is the only one minimal element in the poset* $(\mathcal{U}_{n,\ell}, \preceq)$.

However, this conjecture is, in general, not true. Let G_1 and G_2 be the following two unicyclic graphs of orders 10 (see Fig. 2).



G_1 G_2

Figure 2. Counter-example

Then their Laplacian characteristic polynomials are

$$\begin{aligned}\mathcal{L}(G_1, x) &= x^{10} - 20x^9 + 167x^8 - 758x^7 + 2036x^6 - 3296x^5 + 3130x^4 - 1612x^3 \\ &\quad + 382x^2 - 30x, \\ \mathcal{L}(G_2, x) &= x^{10} - 20x^9 + 168x^8 - 770x^7 + 2091x^6 - 3414x^5 + 3243x^4 - 1642x^3 \\ &\quad + 373x^2 - 30x.\end{aligned}$$

Hence G_1 does not have minimal Laplacian coefficients c_k, $k = 0, \ldots, 10$. So this conjecture is, in general, not true. But it is proven that G_1 is still a minimal element in the poset $\mathcal{U}_{10,2}$. In fact, there are many minimal elements in the poset $\mathcal{U}_{n,\ell}$. This paper is organized as follows: in Section 2, we investigate some properties of minimal elements in the poset $\mathcal{U}_{n,\ell}^g$. In Sections 3 and 4, all minimal elements in four special posets of $\mathcal{U}^g_{n,\ell}$ are characterized, respectively. Finally, in Section 5, we give some conjectures.

2. THE MINIMAL ELEMENTS IN $(\mathcal{U}^G_{N,\ell}, \preceq)$

Let v be a vertex of a connected graph G and let $N_G(v)$ denote the set of the neighbors of v in G. Let $d_G(v)$ denote the degree of v in G, if $d_G(v) = 1$, v is called a leaf or a pendent vertex. Say that $P = vv_1 \cdots v_k$ is a pendant path of length k attached at vertex v if its interval vertices $v_1 \cdots v_{k-1}$ have degree two and v_k is a leaf. If $k = 1$, then v_1 is a *leaf* and vv_1 is called a *pendent edge*. A *branch vertex* is a vertex having degree more than two. Moreover, let $d(u, v)$ denote the distance between vertices u and v. For a n-vertex unicyclic graph $G \in \mathcal{U}^g_{n,\ell}$, G can be obtained from a cycle $C_g = u_1 \cdots u_g$ of order g by attaching trees $T_1 \cdots T_g$ rooted at $u_1, \ldots, u_g$, respectively. So G may be written to be $C_{T_1,\ldots,T_g}$.

Lemma 2.4. [**5**] *Let v be a vertex of a nontrivial connected graph G and for nonnegative integers p and q, let $G(p, q)$ denote the graph obtained from G by adding two pendent paths of lengths p and q at v, respectively, $p \geq q \geq 1$. Then $c_k(G(p, q)) \leq c_k(G(p + 1, q - 1)), k = 0, 1, \ldots, n$.*

Definition 2.5. *Let G be an arbitrary connected graph with n vertices. If uv is a non-pendent cut edge of G with $d_G(v) \geq 3$ and $d_G(u) \geq 2$ such that there's at least one pendent path $P_{t+1} = vv_1 \cdots v_t$ attached at v, then the graph $G' = \xi(G, uv)$ obtained from G by changing all edges (except uv, vv_1) incident with v into new edges between u and $N_G(v) \setminus \{u, v_1\}$. In other words,*

$$G' = G - \{vx | x \in N_G(v) \setminus \{u, v_1\}\} + \{ux | x \in N_G(v) \setminus \{u, v_1\}\}.$$

We say that G' is a ξ-transformation of G. (See Fig. 3).

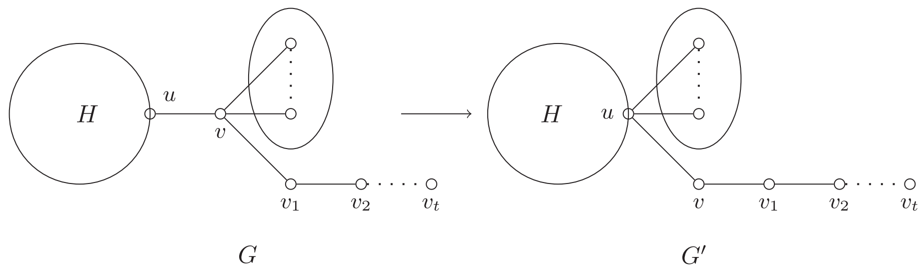


Figure 3. ξ-transformation

Clearly, ξ-transformation preserves the number of leaves in G.

Lemma 2.6. *Let G be a connected graph of order n and G' be obtained from G by ξ-transformation. If there exists a path $P_{s+1} = uu^p_1 \cdots u^p_{s-1}u^p_s$ of order $s + 1$ in the component of $G - uv$ and a pendent path $P_{t+1} = vv_1 \cdots v_t$ attached at v with $s \geq t$, then $G' \preceq G$, i.e,*

$$c_k(G) \geq c_k(G'), k = 0, 1, \ldots, n,$$

with equality if and only if $k \in \{0, 1, n - 1, n\}$.

Proof. Clearly, if $k \in \{0, 1, n\}$, $c_k(G) = c_k(G')$. Since uv is a cut edge, then every spanning tree of G and G' includes edge uv, which implies $\tau(G) = \tau(G')$. Hence $c_{n-1}(G) = c_{n-1}(G')$. Now assume that $2 \le k \le n-2$ and consider the coefficients $c_{n-k}(G)$. Let $\mathcal{F}'$ (resp. $\mathcal{F}$) be the set of all spanning forests of G' (resp. G) with exactly k components. For an arbitrary spanning forest $F' \in \mathcal{F}'$, denote by T' the component of F' containing u. Let $f : \mathcal{F}' \to \mathcal{F}, F = f(F')$, where $V(F) = V(F')$ and

$$E(F) = E(F') - \{ux | x \in N_{T'}(u) \cap N_G(v)\} + \{vx | x \in N_{T'}(u) \cap N_G(v)\}.$$

Then f is injective. Let $\mathcal{F}' = \mathcal{F}'_{(1)} \bigcup \mathcal{F}'_{(2)}$, where $\mathcal{F}'_{(1)} = \{F' \in \mathcal{F}' \mid uv \in E(F')\}$ and $\mathcal{F}'_{(2)} = \{F' \in \mathcal{F}' \mid uv \notin E(F')\}$. If $F' \in \mathcal{F}'_{(1)}$, then F' and $F = f(F')$ have the same components except T'. Moreover, T' and $f(T')$ have the same vertices. Hence $\gamma(F) = \gamma(F')$. If $F' \in \mathcal{F}'_{(2)}$, let S' be the component of F' containing v. Then F' and $F = f(F')$ have the same components except T' and S' in F'. Assume T' contains a vertices in the component of $G - uv$ containing v, $|V(T')| - a$ vertices in the component of $G - uv$ containing u. Then F has two components $f(T') = T$ with $|V(T')| - a$ vertices and $f(S') = S$ with $a + |V(S')|$ vertices corresponding to T' and S', respectively. Denote by N the product of the orders of all components of F' except T' and S'. Then

$$\begin{aligned}\gamma(f(F')) - \gamma(F') &= [(|V(T')| - a)(a + |V(S')|) - |V(T')| \cdot |V(S')|]N \\ &= (|V(T')| - a - |V(S')|) \cdot a \cdot N.\end{aligned}$$

Further let $\mathcal{F}'_{(2)} = \mathcal{F}'_{20} \cup \mathcal{F}'_{21} \cup \mathcal{F}'_{22}$, where

$$\begin{aligned}\mathcal{F}'_{20} &= \{F' \in \mathcal{F}'_{(2)} \mid |V(T')| - a = |V(S')| \text{ or } a = 0\}, \\ \mathcal{F}'_{21} &= \{F' \in \mathcal{F}'_{(2)} \mid |V(T')| - a < |V(S')|, a > 0\}, \\ \mathcal{F}'_{22} &= \{F' \in \mathcal{F}'_{(2)} \mid |V(T')| - a > |V(S')|, a > 0\}.\end{aligned}$$

Hence it follows that

$$\begin{aligned}&\forall F' \in \mathcal{F}'_{20}, \gamma(f(F')) - \gamma(F') = 0, \\ &\forall F' \in \mathcal{F}'_{21}, \gamma(f(F')) - \gamma(F') < 0, \\ &\forall F' \in \mathcal{F}'_{22}, \gamma(f(F')) - \gamma(F') > 0.\end{aligned}$$

For every spanning forest $F'_1 \in \mathcal{F}'_{21}$, let T' and S' be two components of F'_1 containing u and v, respectively. Assume that $u, u_1^p, \ldots, u_{r-1}^p \in V(T')$ and $u_r^p \notin V(T')$. Moreover, let R' be a component of F'_1 containing u_r^p with b vertices. Thus let T'' be a tree obtained from T' and R' by joining u_{r-1}^p and u_r^p with the edge $u_{r-1}^p u_r^p$, S'' be the path $vv_1 \cdots v_{|V(T')|-a-1}$ and R'' be the path $v_{|V(T')|-a} \cdots v_{|V(S')|-1}$. Then $F'_2 = (F'_1 - \{T', S', R'\}) \cup \{T'', S'', R''\}$ is a spanning forest of G' with exactly k components and $F'_2 \in \mathcal{F}'_{22}$. Hence there exists an injective (not bijective) map from $\mathcal{F}'_{21}$ to $\mathcal{F}'_{22}$, i.e.,

$$\varphi : \mathcal{F}'_{21} \to \mathcal{F}'_{22} : F'_1 \to F'_2 = \varphi(F'_1),$$

where $F_2' = \varphi(F_1') = (F_1' - \{T', S', R'\}) \cup \{T'', S'', R''\}$. Note that $|V(T'')| = |V(T')| + b$, $|V(S'')| = |V(T')| - a$, and $|V(R'')| = |V(S')| - |V(T')| + a$. It is easy to see that for $F' \in \mathcal{F}_{21}'$,

$$\begin{aligned}
&\gamma(f(\varphi(F'))) - \gamma(\varphi(F')) \\
&\qquad = \big[(|V(T')| + b - a) \cdot (|V(T')| - a + a) - (|V(T')| + b) \cdot (|V(T')| - a)\big] \\
&\qquad\qquad \times (|V(S')| - |V(T')| + a) \cdot \frac{N}{b} \\
&\qquad = -(|V(T')| - a - |V(S')|) \cdot aN \\
&\qquad = -(\gamma(f(F')) - \gamma(F'))
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{F' \in \mathcal{F}_{21}' \cup \mathcal{F}_{22}'} [\gamma(f(F')) - \gamma(F')] &= \sum_{F' \in \mathcal{F}_{21}'} [\gamma(f(F')) - \gamma(F') + \gamma(f(\varphi(F'))) - \gamma(\varphi(F'))] \\
&+ \sum_{F' \in \mathcal{F}_{22}' \setminus \varphi(\mathcal{F}_{21}')} [\gamma(f(F')) - \gamma(F')] = \sum_{F' \in \mathcal{F}_{22}' \setminus \varphi(\mathcal{F}_{21}')} [\gamma(f(F')) - \gamma(F')] > 0.
\end{aligned}$$

It follows from Theorem 1.1 that

$$c_{n-k}(G') = \sum_{F' \in \mathcal{F}'} \gamma(F') < \sum_{F' \in \mathcal{F}'} \gamma(f(F')) \leq \sum_{F \in \mathcal{F}} \gamma(F) = c_{n-k}(G).$$

So the assertion holds. □

Now we are ready to present the main result in this section.

Theorem 2.7. *If $C_{T_1,\dots,T_g} \in \mathcal{U}_{n,\ell}^g$ is obtained from the cycle $C_g = u_1 \cdots u_g$ by attaching g trees $T_1, \dots, T_g$ at the roots $u_1, \dots, u_g$, respectively, where $|V(T_i)| = n_i$ and the number of leaves in T_i is ℓ_i, then*

$$C_{B_1,\dots,B_g} \preceq C_{T_1,\dots,T_g},$$

where B_i is a tree with root u_i obtained by identifying one end u_{i,p_i} of the path $P_{p_i} : u_i u_{i,2} \cdots u_{i,p_i}$ and the center of $BST_{n_i - p_i + 1, \ell_i}$ for $i = 1, \dots, g$ and $p_1, \dots, p_g$ are equal to 1 except at most one p_j. Moreover, the equality holds if and only if $C_{B_1,\dots,B_g} \cong C_{T_1,\dots,T_g}$.

Proof. Denote $G = C_{T_1,\dots,T_g}$. We prove this result in two steps.

Step 1. Let P be the longest path among all paths which start at u_i in T_i for $i = 1, \dots, g$. Without loss of generality, assume that P belongs to T_1. Let v be the farthest branch vertex from vertex u_i in T_i for $i = 2, \dots, g$. Thus there exists a cut edge uv in T_i such that $d(u_i, u) + 1 = d(u_i, v)$. By ξ-transformation on uv and Lemma 2.6, we obtain $G_1 = \xi(G, uv) = C_{T_1, T_2, \dots, T_i', \dots, T_g} \preceq G$ such that the number of branch vertices in T_i is non-increasing. Hence after a series of ξ-transformations on the cut edge xy with y being a branch vertex and $d(u_i, x) + 1 = d(u_i, y)$ in T_i',

$G_2 = C_{T_1,T_2,\ldots,T_i^{\prime\prime},\ldots,T_g}$ is obtained, where $T_i^{\prime\prime}$ is a starlike tree of order n_i and leaves ℓ_i with root u_i. Hence for $i = 2,\ldots,g$, after performing a series of ξ-transformations and repeatedly using Lemma 2.4, there exists a unicyclic graph $G_3 = C_{T_1,B_2,\ldots,B_g}$ such that $G_3 \preceq G_1$ where B_i is BST_{n_i,ℓ_i} with a center vertex u_i for $i = 2,\ldots,g$.

Step 2. If T_1 is a path or a starlike tree of order n_1, then the assertion holds by using Lemma 2.4. Assume that T_1 has at least one branch vertex except the root u_1. Let v be the branch vertex which is nearest to some pendent vertices in T_1 (maybe v is not unique). Then there exists a cut edge uv, $d(u_1,u)+1 = d(u_1,v)$. If there exists uv as defined above which satisfies the conditions of Lemma 2.6, then $G_4 = \xi(G_3,uv) \preceq G_3$. Moreover, the number of branch vertices in G_4 is no more than that in G_3. After performing a series of this type of ξ-transformations and repeatedly using Lemma 2.4, there exists an unicyclic graph $G_5 = C_{T_1',B_2,\ldots,B_g}$ such that $G_5 \preceq G_4$, where T_1' is a tree rooted at u_1 obtained by attaching $\ell_1 - 1$ pendent paths at some vertices of the longest path P' of T_1'. If all pendent paths are attached at the only one vertex u_1 or $u_{1,x}$ of P', then the result holds.

Otherwise, let v' be the branch vertex which is nearest to u_1 in T_1' (when $d_{G_5}(u_1) > 3$, $v' = u_1$). Then there exists a cut edge $u'v'$ which satisfies $d(u_1,v')+1 = d(u_1,u')$. By ξ-transformation on $u'v'$ and Lemma 2.6, we obtain $G_6 = \xi(G_5,u'v') \preceq G_5$. Further, the number of branch vertices in G_6 is no more than that in G_5. Hence by performing a series of this type of ξ-transformations, $G_7 = C_{B_1,\ldots,B_g}$ is obtained, where B_1 has exactly one branch vertex u_1 or B_1 has exactly one branch vertex $u \neq u_1$ with $d_{G_7}(u_1) = 3$. If $u = u_1$, then by Lemma 2.4, the assertion holds. If $u \neq u_1$ and $d_{G_7}(u_1) = 3$, then applying Lemma 2.4 to all pendent paths in G_7 yields the result.

Corollary 2.8. *Let $C_{T_1,\ldots,T_g} \in \mathcal{U}_{n,\ell}^g$ be obtained from the cycle $C_g = u_1\cdots u_g$ by attaching g trees $T_1,\ldots,T_g$ at the roots $u_1,\ldots,u_g$, respectively. If $d(u_1) > 3$ and $|V(T_i)| = 1$ for $i = 2,\ldots,g$, then*

$$C_{B_1,\ldots,B_g} \preceq C_{T_1,\ldots,T_g},$$

where B_1 is $BST_{n-g+1,\ell}$ with a center vertex u_1 and $|V(B_i)| = 1$ for $i = 2,\ldots,g$. Moreover, the equality holds if and only if $C_{B_1,\ldots,B_g} \cong C_{T_1,\ldots,T_g}$.

Proof. It is obvious that the assertion follows from the proof of Theorem 2.7.

3. THE MINIMAL ELEMENTS IN TWO SUBSETS OF $\mathcal{U}_{N,\ell}^G$

In this section, we characterize all extremal graphs which have minimal Laplacian coefficients in the following two special subsets of $\mathcal{U}_{n,\ell}^g$. Denote

$$\mathcal{U}_{n,\ell}^{g,1} = \{C_{T_1,\ldots,T_g} \mid |V(T_i)| = 1 \text{ for } i = 2,\ldots,g\},$$
$$\mathcal{U}_{n,\ell}^{g,2} = \{C_{T_1,\ldots,T_g} \mid |V(T_1)| > 1, |V(T_i)| > 1, |V(T_j)| = 1 \text{ for } j \neq 1,i\}.$$

Clearly, $U_{n,\ell}^{g,p}$ is in $\mathcal{U}_{n,\ell}^{g,1}$. For convenience, denote $U_{n,\ell}^{g,0} = U^0, U_{n,\ell}^{g,1} = U^1,\ldots,U_{n,\ell}^{g,p} = U^p$.

Lemma 3.9. *For* $1 \leq p \leq \left\lfloor \dfrac{n-g-g\ell+\ell}{\ell+1} \right\rfloor$, U^p *and* U^{p-1} *are incomparable in the poset* $(\mathcal{U}^g_{n,\ell}, \preceq)$. (See Fig. 1).

Proof. We first show that $c_{n-2}(U^p) < c_{n-2}(U^{p-1})$. It is obvious that U^{p-1} can be regarded as $U^{p-1} = \xi(U^p, w_{p-1}w_p)$. For convenience, we denote U^p and U^{p-1} by G and G', respectively. Let $\mathcal{F}_2$ (resp. $\mathcal{F}'_2$) be the set of all spanning forests of G (resp. G') with exactly 2 components. For an arbitrary spanning forest $F \in \mathcal{F}_2$ (resp. $F' \in \mathcal{F}'_2$), F (resp. F') can be obtained by deleting two edges $\{e_1, e_2\}$ in $E(G)$ (resp. $E(G')$) with e_1 belonging to the cycle in G (resp. G'). If $e_2 \neq w_{p-1}w_p$, then F and F' have the same components, which implies $\gamma(F) = \gamma(F')$. If $e_2 = w_{p-1}w_p$, then

$$\begin{aligned}
&\gamma(F) - \gamma(F') \\
&= (g+p-1)(n-g-p+1) - \left(\left\lfloor \frac{n-g-p}{\ell} \right\rfloor + 1\right)\left(n - \left\lfloor \frac{n-g-p}{\ell} \right\rfloor - 1\right) < 0,
\end{aligned}$$

Therefore,

$$c_{n-2}(G) - c_{n-2}(G') = \sum_{F \in \mathcal{F}_2} \gamma(F) - \sum_{F' \in \mathcal{F}'_2} \gamma(F') < 0.$$

We next show $c_m(G) > c_m(G')$ for $2p \leq m \leq 2(p+g)-3$. Clearly $P_{p+g-1} = w_{p-1}w_{p-2}\cdots w_1u_1u_2\cdots u_g$ (denote $w_i = u_{-i+1}, 1 \leq i \leq p-1$ for convenience) is the longest path of order $p+g-1$ in the component of $G - w_{p-1}w_p$. Let $\mathcal{F}'$ (resp. $\mathcal{F}$) be the set of all spanning forests of G' (resp. G) with exactly $n-m$ components, in other words, $\mathcal{F}'$ (resp. $\mathcal{F}$) is the set of all spanning forests of G' (resp. G) with exactly m edges. For an arbitrary spanning forest $F' \in \mathcal{F}'$, denote by T' the component of F' containing w_{p-1}. Let $f : \mathcal{F}' \to \mathcal{F}, F = f(F')$, where $V(F) = V(F')$ and

$$\begin{aligned}
E(F) = E(F') &- \{w_{p-1}x | x \in N_{T'}(w_{p-1}) \cap N_G(w_p)\} \\
&+ \{w_p x | x \in N_{T'}(w_{p-1}) \cap N_G(w_p)\},
\end{aligned}$$

then f is injective. Let $\mathcal{F}' = \mathcal{F}'_{(1)} \bigcup \mathcal{F}'_{(2)}$, where $\mathcal{F}'_{(1)} = \{F' \in \mathcal{F}' \mid w_{p-1}w_p \in E(F')\}$ and $\mathcal{F}'_{(2)} = \{F' \in \mathcal{F}' \mid w_{p-1}w_p \notin E(F')\}$. If $F' \in \mathcal{F}'_{(1)}$, then F' and $F = f(F')$ have the same components except T'. Moreover, T' and $f(T')$ have the same vertices. Hence $\gamma(F) = \gamma(F')$. If $F' \in \mathcal{F}'_{(2)}$, let S' be the component of F' containing w_p. Then F' and $F = f(F')$ have the same components except T' and S' in F'. Assume that T' contains a vertices in the component of $G - w_{p-1}w_p$ containing w_p, $|V(T')| - a$ vertices in the component of $G - w_{p-1}w_p$ containing w_{p-1}. Then F has two components $f(T') = T$ with $|V(T')| - a$ vertices and $f(S') = S$ with $a + |V(S')|$ vertices corresponding to T' and S', respectively. Denote by N the product of the orders of all components of F' except T' and S'. Then

$$\begin{aligned}
\gamma(f(F')) - \gamma(F') &= \left[(|V(T')| - a)(a + |V(S')|) - |V(T')| \cdot |V(S')|\right] N \\
&= (|V(T')| - a - |V(S')|) \cdot a \cdot N.
\end{aligned}$$

Further, let $\mathcal{F}'_{(2)} = \mathcal{F}'_{20} \cup \mathcal{F}'_{21} \cup \mathcal{F}'_{22}$, where

$$\begin{aligned}
\mathcal{F}'_{20} &= \{F' \in \mathcal{F}'_{(2)} \mid |V(T')| - a = |V(S')| \text{ or } a = 0\},\\
\mathcal{F}'_{21} &= \{F' \in \mathcal{F}'_{(2)} \mid |V(T')| - a < |V(S')|, a > 0\},\\
\mathcal{F}'_{22} &= \{F' \in \mathcal{F}'_{(2)} \mid |V(T')| - a > |V(S')|, a > 0\}.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
&\forall F' \in \mathcal{F}'_{20}, \gamma(f(F^{'})) - \gamma(F^{'}) = 0,\\
&\forall F' \in \mathcal{F}'_{21}, \gamma(f(F^{'})) - \gamma(F^{'}) < 0,\\
&\forall F' \in \mathcal{F}'_{22}, \gamma(f(F^{'})) - \gamma(F^{'}) > 0.
\end{aligned}$$

For every spanning forest $F'_1 \in \mathcal{F}'_{21}$, let T' and S' be two components of F'_1 containing w_{p-1} and w_p, respectively. Then T' does not contain all the vertices of the path P_{p+g-1}. In fact, if all the vertices of P_{p+g-1} belong to T', then by the definition of $\mathcal{F}'_{21}$, we have $|V(T')| = p + g - 1 + a$, $|E(T')| \geq p + g - 1$, and $|V(S')| > |V(T')| - a = p + g - 1$, $|E(S')| > p + g - 2$, which implies $m \geq |E(T')| + |E(S')| > p + g - 1 + p + g - 2 = 2(p + g) - 3$. It is a contradiction to $m \leq 2(p + g) - 3$. Therefore, assume that $u_{-(p-1)+1}, u_{-(p-2)+1}, \ldots, u_{r-1} \in V(T')$ and $u_r \notin V(T')$. Moreover, let R' be a component of F'_1 containing u_r with b vertices. Thus let T'' be a tree obtained from T' and R' by joining u_{r-1} and u_r with edge $u_{r-1}u_r$, S'' be the path $w_p v_1 \cdots v_{|V(T')|-a-1}$ and R'' be the path $v_{|V(T')|-a} \cdots v_{|V(S')|-1}$. Then $F'_2 = (F'_1 - \{T', S', R'\}) \cup \{T'', S'', R''\}$ is a spanning forest of G' with exactly m edges and $F'_2 \in \mathcal{F}'_{22}$. Hence there exists an injective map from $\mathcal{F}'_{21}$ to $\mathcal{F}'_{22}$, i.e.,

$$\varphi : \mathcal{F}'_{21} \to \mathcal{F}'_{22} : F'_1 \to F'_2 = \varphi(F'_1),$$

where $F'_2 = \varphi(F'_1) = (F'_1 - \{T', S', R'\}) \cup \{T'', S'', R''\}$. Note that $|V(T'')| = |V(T')| + b$, $|V(S'')| = |V(T')| - a$, and $|V(R'')| = |V(S')| - |V(T')| + a$. It is easy to see that for $F' \in \mathcal{F}'_{21}$,

$$\begin{aligned}
&\gamma(f(\varphi(F'))) - \gamma(\varphi(F'))\\
&\qquad= [(|V(T')| + b - a) \cdot (|V(T')| - a + a) - (|V(T')| + b) \cdot (|V(T')| - a)]\\
&\qquad\quad\times (|V(S')| - |V(T')| + a) \cdot \frac{N}{b}\\
&\qquad= -(|V(T')| - a - |V(S')|) \cdot aN\\
&\qquad= -(\gamma(f(F')) - \gamma(F'))
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{F' \in \mathcal{F}'_{21} \cup \mathcal{F}'_{22}} [\gamma(f(F')) - \gamma(F')] = \sum_{F' \in \mathcal{F}'_{21}} [\gamma(f(F')) - \gamma(F') + \gamma(f(\varphi(F'))) - \gamma(\varphi(F'))]\\
&+ \sum_{F' \in \mathcal{F}'_{22} \setminus \varphi(\mathcal{F}'_{21})} [\gamma(f(F')) - \gamma(F')] = \sum_{F' \in \mathcal{F}'_{22} \setminus \varphi(\mathcal{F}'_{21})} [\gamma(f(F')) - \gamma(F')] \geq 0.
\end{aligned}$$

It follows from Theorem 1.1 that for $m \leq 2(p+g)-3$,

$$c_m(G') = \sum_{F' \in \mathcal{F}'} \gamma(F') \leq \sum_{F' \in \mathcal{F}'} \gamma(f(F')) \leq \sum_{F \in \mathcal{F}} \gamma(F) = c_m(G).$$

Further we show that the above inequality is strict for $m \geq 2p$. In other words, we show that φ is not a bijective map for $2p \leq m \leq 2(p+g)-3$. Hence it is sufficient to find a spanning forest $\bar{F}_2' \in \mathcal{F}_{22}'$, such that $\bar{F}_2' \notin \varphi(\mathcal{F}_{21}')$. Let $\bar{T}'', \bar{S}'' \in \bar{F}_2'$, where $\bar{T}''$ is the path: $x_1 w_{p-1} w_{p-2} \cdots w_1 u_1 u_g (x_1 \in N_G(w_p) \setminus \{w_{p-1}, v_1\})$, $\bar{S}''$ is the path: $w_p v_1 \cdots v_{p-1}$. The rest $m-2p$ edges are chosen from the edge set $E(G) \setminus (\{w_{p-1}w_p, u_1u_2, u_gu_{g-1}, v_{p-1}v_p\} \cup E(\bar{T}'') \cup E(\bar{S}''))$ of order $n-2p-4$, it is obvious that $n-2p-4 \geq m-2p$. Suppose that there exists $\bar{F}_1' \in \mathcal{F}_{21}'$, and $\varphi(\bar{F}_1') = \bar{F}_2'$. Let $\bar{T}'$ be the component of $\bar{F}_1'$ which contains w_{p-1}. By the definition of φ, $|V(\bar{S}'')| = |V(\bar{T}')| - a$, so u_2 is the first vertex on P_{p+g-1} which does not belong to $\bar{T}'$, then $u_1u_2 \in \varphi(\bar{F}_1')$. Since $u_1u_2 \notin \bar{F}_2'$, then $\varphi(\bar{F}_1') \neq \bar{F}_2'$, a contradiction. This completes the proof of Lemma 3.9.

Lemma 3.10. *For* $1 \leq p \neq q \leq \left\lfloor \frac{n-g-g\ell+\ell}{\ell+1} \right\rfloor$, U^p *and* U^q *are incomparable in the poset* $(\mathcal{U}_{n,\ell}^{g}, \preceq)$.

Proof. Without loss of generality, assume $p-q=h \geq 1$. By using Lemma 3.9 repeatedly, we have

$$c_{n-2}(U^p) < c_{n-2}(U^{p-1}) < \cdots < c_{n-2}(U^q),$$

$$c_m(U^p) \geq c_m(U^{p-1}) \geq \cdots c_m(U^{q+1}) > c_m(U^q), \text{ for } 2(q+1) \leq m \leq 2(q+g+1)-3.$$

Thus the assertion holds. □

Now we are ready to characterize all minimal elements in $\mathcal{U}_{n,\ell}^{g,1}$.

Theorem 3.11. *There are exactly* $p+1$ *minimal elements* $U^0, \ldots, U^p$ *in* $\mathcal{U}_{n,\ell}^{g,1}$, *where* $p = \left\lfloor \frac{n-g-g\ell+\ell}{\ell+1} \right\rfloor$.

Proof. For any $G = C_{T_1,T_2,\ldots,T_g} \in \mathcal{U}_{n,\ell}^{g,1}$ with $|V(T_i)| = 1$ for $i = 2, \ldots, g$, by Theorem 2.7, there exists a graph $G \succeq G_1 = C_{B_1,B_2,\ldots,B_g} \in \mathcal{U}_{n,\ell}^{g,1}$, where B_1 is a tree with root u_1 obtained by identifying one end u_{1,p_1} of the path P_{p_1} : $u_1 u_{1,2} \cdots u_{1,p_1}$ and the center of $BST_{n_1-p_1+1,l_1}$ and $|V(B_i)| = 1$ for $i = 2, \ldots, g$. If $p_1 > \left\lfloor \frac{n-g-g\ell+\ell}{\ell+1} \right\rfloor$, then by Lemma 2.6, there exists an cut edge uv such that $G_1 \succeq G_2 = \xi(G_1, uv) = C_{B_1',B_2,\ldots,B_g} \in \mathcal{U}_{n,\ell}^{g,1}$, where B_1' is a tree with root u_1 obtained by identifying one end u_{1,p_1-1} of the path $P_{p_1-1} : u_1 u_{1,2} \cdots u_{1,p_1-1}$ and the center of $BST_{n_1-p_1+2,\ell_1}$. After a series of ξ-transformations, there exists a U^p such that $G \succeq U^p$. If $p_1 \leq \left\lfloor \frac{n-g-g\ell+\ell}{\ell+1} \right\rfloor$, then $G \succeq G_1 = U^{p_1}$. On the other hand, by Lemma 3.10, $U^0, \ldots, U^p$ are incomparable in the poset $(\mathcal{U}_{n,\ell}^{g,1}, \preceq)$. Hence $U^0, \ldots, U^p$ are exactly all minimal elements in the poset $(\mathcal{U}_{n,\ell}^{g,1}, \preceq)$.

Theorem 3.12. *For any* $G = C_{T_1,\dots,T_g} \in \mathcal{U}^{g,2}_{n,\ell}$, $U^0 \prec G$.

Proof. By Theorem 2.7, we may assume that $G = C_{B_1,\dots,B_i,\dots,B_g} \in \mathcal{U}^{g,2}_{n,\ell}$, where B_1 is a tree with root u_1 obtained by identifying one end u_{1,p_1} of the path P_{p_1} : $u_1u_{1,2}\cdots u_{1,p_1}$ and the center vertex of $BST_{n_1-p_1+1,\ell_1}$, B_i is BST_{n_i,ℓ_i} with a center vertex u_i, and $|V(B_j)| = 1$ for $j \neq 1, i$. Let

$$G' = G - \{u_ix | x \in N_G(u_i) \setminus \{u_{i+1}, u_{i-1}\}\} + \{u_1x | x \in N_G(u_i) \setminus \{u_{i+1}, u_{i-1}\}\}.$$

We will prove $c_k(G) \geq c_k(G')$ with at least one strict inequality. Clearly, when $k \in \{0, 1, n-1, n\}$, $c_k(G) = c_k(G')$. For $2 \leq k \leq n-2$, let $\mathcal{F}$ and $\mathcal{F}'$ be the sets of all spanning forests of G and G' with exactly k components, respectively. For an arbitrary spanning forest $F' \in \mathcal{F}'$ with T' being the component of F' containing u_1, let $f : \mathcal{F}' \to \mathcal{F}, F' \to F = f(F')$, where $V(F) = V(F')$, and

$$\begin{aligned} E(F) = E(F') &- \{u_1x | x \in N_G(u_i) \cap N_{T'}(u_1) \setminus V(C_g)\} \\ &+ \{u_ix | x \in N_G(u_i) \cap N_{T'}(u_1) \setminus V(C_g)\}. \end{aligned}$$

Then f is injective from $\mathcal{F}'$ to $\mathcal{F}$. Denote by N the product of the orders of all components containing no u_1, u_i. If $u_i \in T'$, then F' and F have the same components except for T'. Moreover, F has a component containing u_1 which corresponds to T' in F'. Clearly, the two components have the same orders. Hence $\gamma(F) = \gamma(F')$. If $u_i \notin T'$, assume u_i is in a component S' of F'. Moreover, there are $b \geq 0$ vertices in the connected component containing u_2 in $T' - u_1u_2$, and $d \geq 0$ vertices in the connected component containing u_g in $T' - u_1u_g$, $e_1 \geq 1$ vertices (including u_1) in the vertex set $V(T_1)$ and $e_2 \geq 0$ vertices in the vertex set $V(T_i) \setminus \{u_i\}$. Furthermore, the tree S' contains $a \geq 1$ vertices in the connected component containing u_i in $S' - u_iu_{i+1}$ and $c \geq 0$ vertices in the connected component containing u_{i+1} in $S' - u_iu_{i+1}$. Then F' and F have the same components except for T' and S'. Moreover, F have two trees T containing u_1 and S containing u_i which correspond to T' and S' in F', respectively. Hence

$$\begin{aligned} \gamma(f(F')) - \gamma(F') &= [(a + c + e_2)(b + d + e_1) - (a + c)(b + d + e_1 + e_2)]N \\ &= e_2(b + d + e_1 - a - c)N. \end{aligned}$$

Consider the subset $\bar{\mathcal{F}}'$ of those spanning forests F' with k components which coincide on $G' \backslash (T' \cup S')$ with fixed values e_1, e_2. Since the two parts of F' on the cycle between T' and S' may be translated, let $a + b = M_1, c + d = M_2$ be fixed. Then

$$\begin{aligned} &\sum_{\substack{F' \in \bar{\mathcal{F}}',\ a+b=M_1 \\ c+d=M_2}} \big(\gamma(f(F')) - \gamma(F')\big) = \sum_{\substack{a+b=M_1 \\ c+d=M_2}} e_2(b + d + e_1 - a - c)N \\ &= e_2N \sum_{c=0}^{M_2} \sum_{b=0}^{M_1-1} (2b + M_2 + e_1 - 2c - M_1) = e_2NM_1 \sum_{c=0}^{M_2} (e_1 + M_2 - 2c - 1) \\ &= e_2NM_1(e_1 - 1)(M_2 + 1) \geq 0. \end{aligned}$$

Hence

$$\sum_{F'\in\mathcal{F}'}(\gamma(f(F'))-\gamma(F'))=\sum_{e_1}\sum_{e_2}\sum_{M_1}\sum_{M_2}\sum_{\substack{F'\in\bar{\mathcal{F}}',\ a+b=M_1\\ c+d=M_2}}(\gamma(f(F'))-\gamma(F'))\geq 0.$$

Since $|V(T_1)|>1, |V(T_i)|>1$, there exists one forest F' such that $e_1>1$ and $e_2>0$. Therefore

$$c_{n-k}(G')=\sum_{F'\in\mathcal{F}'}\gamma(F')<\sum_{F'\in\mathcal{F}'}\gamma(f(F'))\leq\sum_{F\in\mathcal{F}}\gamma(F)=c_{n-k}(G), 2\leq k\leq n-2.$$

Hence by Corollary 2.8, we have $U^0\preceq G'$ with equality if and only if $G'\cong U^0$. Therefore, $U^0\preceq G'\preceq G$, the assertion holds. □

It follows from Theorems 1.2, 3.11 and 3.12 that the following results hold.

Corollary 3.13. *Let $G=C_{T_1,\ldots,T_g}$ be an arbitrary unicyclic graph in $\mathcal{U}^{g,1}_{n,\ell}$. Then for $p=\left\lfloor\dfrac{n-g-g\ell+\ell}{\ell+1}\right\rfloor$,*

$$LEL(G)\geq\min\{LEL(U^0), LEL(U^1),\ldots, LEL(U^p)\}.$$

Corollary 3.14. *Let $G=C_{T_1,\ldots,T_g}$ be an arbitrary unicyclic graph in $\mathcal{U}^{g,2}_{n,\ell}$. Then $LEL(G)>LEL(U^0)$.*

4. THE MINIMAL ELEMENTS IN $\mathcal{U}^3_{N,\ell}$ AND $\mathcal{U}^4_{N,\ell}$

In this section, we determine all the minimal elements in the posets $(\mathcal{U}^3_{n,\ell},\preceq)$ and $(\mathcal{U}^4_{n,\ell},\preceq)$. Before stating our results, we need the following definitions.

Definition 4.15. *For any $G\in\mathcal{U}^3_{n,\ell}$, let G^* be the graph obtained from G by changing all the edges (except $E(C_3)$) incident with u_2, u_3 into new edges between u_1 and $N_G(u_2)\cup N_G(u_3)\setminus V(C_3)$. In other words,*

$$G^*=G-\{u_2x|x\in N_G(u_2)\setminus V(C_3)\}-\{u_3x|x\in N_G(u_3)\setminus V(C_3)\}$$
$$+\{u_1x|x\in N_G(u_2)\cup N_G(u_3)\setminus V(C_3)\}.$$

We say G^* is a η-transformation of G. (See Fig. 4).

Performing this transformation to the graph G can be seen as performing the transformation of Lemma 1 in [**13**] to G twice.

Lemma 4.16. *For $G\in\mathcal{U}^3_{n,\ell}$, if G^* is obtained from G by η-transformation, then $G^*\preceq G$, i.e., $c_k(G^*)\leq c_k(G)$ with equality if and only if $k\in\{0,1,n-1,n\}$.*

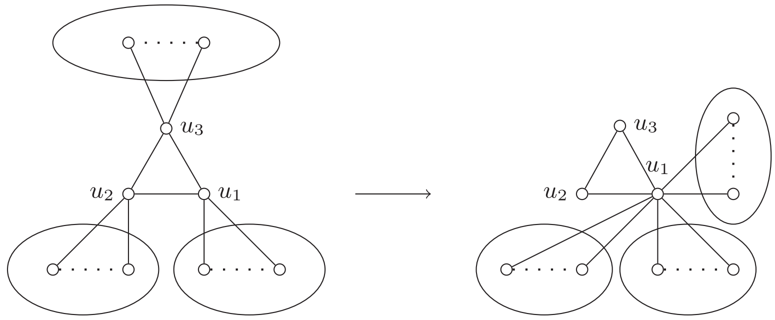


Figure 4. η-transformation

Proof. Clearly, when $k \in \{0,1,n-1,n\}$, $c_k(G) = c_k(G^*)$. For $2 \le k \le n-2$, let $\mathcal{F}'$ (resp. $\mathcal{F}$) be the set of all spanning forests of G^* (resp. G) with exactly $n-k$ components. Let $\mathcal{F}' = \mathcal{F}'^{(1)} \cup \mathcal{F}'^{(2)} \cup \mathcal{F}'^{(3)}$, where $\mathcal{F}'^{(j)}, j = 1,2,3$ is the set of all spanning forests of G^* in which u_1, u_2, u_3 belong to exactly j different components. Similarly, $\mathcal{F}^{(j)}, j = 1,2,3$ can be defined. Let $f : \mathcal{F}' \to \mathcal{F}$ with $F' \to F = f(F')$, where $V(F) = V(F')$ and

$$\begin{aligned} E(F) = E(F') &- \{u_1x | x \in N_{R'}(u_1) \cap N_G(u_2) \setminus \{u_3\}\} \\ &- \{u_1x | x \in N_{R'}(u_1) \cap N_G(u_3) \setminus \{u_2\}\} \\ &+ \{u_2x | x \in N_{R'}(u_1) \cap N_G(u_2) \setminus \{u_3\}\} \\ &+ \{u_3x | x \in N_{R'}(u_1) \cap N_G(u_3) \setminus \{u_2\}\}, \end{aligned}$$

for R' being a component of F' containing u_1. Clearly f is injective and $f(\mathcal{F}'^{(j)}) \subseteq \mathcal{F}^{(j)}$ for $j = 1,2,3$. Denote $|V(T_1) \cap V(R') \setminus \{u_1\}| = a, |V(T_2) \cap V(R') \setminus \{u_2\}| = b, |V(T_3) \cap V(R') \setminus \{u_3\}| = c$, where $a,b,c \ge 0$. Let N be the product of the orders of all components of F' containing no $\{u_1,u_2,u_3\}$. Now we consider the following three cases.

Case 1. $F' \in \mathcal{F}'^{(1)}$, u_1,u_2,u_3 belong to one component, then $\gamma(F) = \gamma(F')$. Thus

$$\sum_{F' \in \mathcal{F}'^{(1)}} [\gamma(F) - \gamma(F')] = 0.$$

Case 2. $F' \in \mathcal{F}'^{(2)}$, u_1,u_2,u_3 are in two components, then $\gamma(F) - \gamma(F') = [(a+b+2)(c+1)+(a+c+2)(b+1)+(b+c+2)(a+1)-2(a+b+c+1)-2(a+b+c+2)]N = [(a+b)c + (a+c)b + (b+c)a]N \ge 0$. Thus

$$\sum_{F' \in \mathcal{F}'^{(2)}} [\gamma(F) - \gamma(F')] \ge 0.$$

Case 3. $F' \in \mathcal{F}'^{(3)}$, u_1,u_2,u_3 are in three components, then $\gamma(F) - \gamma(F') = [(a+1)(b+1)(c+1) - (a+b+c+1)]N = (abc + ab + ac + bc)N \ge 0$. Thus

$$\sum_{F' \in \mathcal{F}'^{(3)}} [\gamma(F) - \gamma(F')] \ge 0.$$

Now the inequality $c_k(G^*) < c_k(G), k = 2, 3, \ldots, n-2$ holds from Theorem 1.1 by summing over all possible subsets $\mathcal{F}'$ of spanning forests F' of G^* with $n-k$ components.

Theorem 4.17. *There are exactly* $p+1$ *minimal elements* $U_{n,\ell}^{3,0}, U_{n,\ell}^{3,1} \ldots, U_{n,\ell}^{3,p}$ *in the poset* $(\mathcal{U}_{n,\ell}^3, \preceq)$, *where* $p = \left\lfloor \dfrac{n-3-2\ell}{\ell+1} \right\rfloor$.

Proof. The assertion follows from Lemma 4.16 and Theorems 3.11 and 3.12.

Definition 4.18. *For* $G \in \mathcal{U}_{n,\ell}^4$, *let* $G^\star$ *be the graph obtained from* G *by changing all the edges (except* $E(C_4)$*) incident with* u_2, u_3, u_4 *into new edges between* u_1 *and* $\cup_{i=2}^4 N_G(u_i) \setminus V(C_4)$. *In other words,*

$$\begin{aligned} G^\star &= G - \{u_2x | x \in N_G(u_2) \setminus V(C_4)\} - \{u_3x | x \in N_G(u_3) \setminus V(C_4)\} \\ &\quad - \{u_4x | x \in N_G(u_4) \setminus V(C_4)\} + \{u_1x | x \in \cup_{i=2}^4 N_G(u_i) \setminus V(C_4)\}. \end{aligned}$$

We say $G^\star$ *is a* κ*-transformation of* G.

Since for bipartite graphs, the Laplacian coefficients and the signless Laplacian coefficients are equal, from Lemma 3.1 in [**18**], we have the following lemma:

Lemma 4.19. *For* $G \in \mathcal{U}_{n,\ell}^4$, *if* $G^\star$ *is obtained from* G *by* κ*-transformation, then* $G^\star \preceq G$, *i.e.,* $c_k(G^\star) \leq c_k(G)$ *with equality if and only if* $k \in \{0, 1, n-1, n\}$.

By combining Lemma 4.19 and Theorem 3.11, we have the following theorem.

Theorem 4.20. *There are exactly* $p+1$ *minimal elements* $U_{n,\ell}^{4,0}, U_{n,\ell}^{4,1} \ldots, U_{n,\ell}^{4,p}$ *in the poset* $(\mathcal{U}_{n,\ell}^4, \preceq)$, *where* $p = \left\lfloor \dfrac{n-4-3\ell}{\ell+1} \right\rfloor$.

From Theorem 1.2, we have the following corollary:

Corollary 4.21. (1). *Let* $G = C_{T_1,T_2,T_3} \in \mathcal{U}_{n,\ell}^3$. *Then for* $p = \left\lfloor \dfrac{n-3-2\ell}{\ell+1} \right\rfloor$,

$$LEL(G) \geq \min\{U_{n,\ell}^{3,0}, U_{n,\ell}^{3,1} \ldots, U_{n,\ell}^{3,p}\}.$$

(2). *Let* $G = C_{T_1,T_2,T_3,T_4} \in \mathcal{U}_{n,\ell}^4$. *Then for* $p = \left\lfloor \dfrac{n-4-3\ell}{\ell+1} \right\rfloor$,

$$LEL(G) \geq \min\{U_{n,\ell}^{4,0}, U_{n,\ell}^{4,1} \ldots, U_{n,\ell}^{4,p}\}.$$

5. REMARKS

Although we showed that Ilić and Ilić's conjecture does not hold, it might be true in a slightly modified version. In fact, if there are at least two vertices in the cycle with degrees at least 3, then the conjecture is true for $g = 3$ and $g = 4$. Moreover, we compile a computer program with Matlab software and check that the conjecture is still true for all unicyclic graphs on ≤ 30 vertices with fixed ℓ, g and having at least three vertices in the cycle with degrees at least 3. Hence their conjecture can be modified as follows:

Conjecture 5.22. (1). *For $G \in \mathcal{U}^g_{n,\ell}$, if there are more than two vertices in the cycle having degrees* ≥ 3, *then* $U^0 \preceq G$, *with equality if and only if* $G \cong U^0$.

(2). *There are exactly* $p+1$ *minimal elements* $U^{3,0}_{n,\ell}, U^{3,1}_{n,\ell}, \ldots, U^{3,p}_{n,\ell}$ *in the poset* $(\mathcal{U}_{n,\ell}, \preceq)$, *where* $p = \left\lfloor \dfrac{n-3-2\ell}{\ell+1} \right\rfloor$.

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