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THE LINEAR $(n-1)$-ARBORICITY OF CARTESIAN PRODUCT GRAPHS

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A linear k-forest of an undirected graph G is a subgraph of G whose components are paths with lengths at most k. The linear k-arboricity of G, denoted by $\mathrm{la}_k(G)$, is the minimum number of linear k-forests needed to partition the edge set $E(G)$ of G. In this paper, the exact values of the linear $(n-1)$-arboricity of Hamming graph, and Cartesian product graphs C_{nt}^m and $K_n \square K_{n,n}$ are obtained.

1. INTRODUCTION

Throughout this paper, all graphs considered are finite, undirected, and simple. Let $\mathbb{N}$ represent the set of natural numbers and $[a, b]$ denote the set $\{n \in \mathbb{N} | a \leq n \leq b\}$ for $a \leq b$. A *decomposition* of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. If a graph G has a decomposition $G_1, G_2, \ldots, G_d$, then we say that $G_1, G_2, \ldots, G_d$ decompose G, or G can be decomposed into $G_1, G_2, \ldots, G_d$. Furthermore, a *linear k-forest* is a forest whose components are paths of length at most k. The *linear k-arboricity* of a graph G, denoted by $\mathrm{la}_k(G)$, is the least number of linear k-forests needed to decompose G. We refer to [**18**] for other terminology in graph theory.

The notion of linear k-arboricity of a graph was first introduced by Habib and Peroche [**13**]. It is a natural generalization of edge coloring. Clearly, a linear 1-forest is induced by a *matching*, and $\mathrm{la}_1(G)$ is the edge chromatic number, or chromatic index, $\chi'(G)$ of a graph. Moreover, the linear k-arboricity $\mathrm{la}_k(G)$ is also a refinement of the ordinary linear arboricity $\mathrm{la}(G)$ (or $\mathrm{la}_\infty(G)$) which is the case when every component of each forest is a path with no length constraint.

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The Cartesian product of m graphs $G_1, G_2, \ldots, G_m$ is the graph

$$H = G_1 \Box G_2 \Box \cdots \Box G_m$$

whose vertex set is $\prod_{i=1}^{m} V(G_i)$ and two vertices $(u_1, u_2, \ldots, u_m)$ and $(v_1, v_2, \ldots, v_m)$ are adjacent if and only if $u_j v_j \in E(G_j)$ for some j and $u_i = v_i$ for all other $i \neq j$. When $G_i = G$ for all i, we use G^m to denote $G_1 \Box G_2 \Box \cdots \Box G_m$. It is not difficult to see that

$$|V(H)| = \prod_{i=1}^{m} |V(G_i)|, \text{ and } d_H(u) = \sum_{j=1}^{m} d_{G_j}(u_j)$$

for any vertex $u = (u_1, u_2, \ldots, u_m)$, and

$$|E(H)| = \sum_{j=1}^{m} \Big[|E(G_j)| \prod_{i \neq j} |V(G_i)| \Big].$$

A Hamming graph is the Cartesian product $K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_m}$ of complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_m}$. Graph products are interesting and useful in many situations. For example, in [**16**], Sabidussi showed that any graph has the unique decomposition into prime factors under the Cartesian product. The complexity of many problems that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complicated prime factors.

In 1982, Habib and Peroche proposed the following conjecture for an upper bound on $\mathrm{la}_k(G)$.

Conjecture 1.1. [**12**] *If G is a graph with maximum degree $\Delta(G)$ and $k \geq 2$, then*

$$\mathrm{la}_k(G) \leq \begin{cases} \left\lceil \dfrac{\Delta(G)\cdot \mid V(G) \mid}{2\lfloor \frac{k\cdot|V(G)|}{k+1} \rfloor} \right\rceil, & \textit{when } \Delta(G) = \mid V(G) \mid -1, \\ \left\lceil \dfrac{\Delta(G)\cdot \mid V(G) \mid +1}{2\lfloor \frac{k\cdot|V(G)|}{k+1} \rfloor} \right\rceil, & \textit{when } \Delta(G) < \mid V(G) \mid -1. \end{cases}$$

For $k = \mid V(G) \mid -1$, it is Akiyama's conjecture.

Conjecture 1.2. [**2**] $\mathrm{la}(G) \leq \left\lceil \dfrac{\Delta(G)+1}{2} \right\rceil$.

So far, quite a few results on the verification of Conjecture 1.1 have been obtained in the literature, especially for graphs with particular structures, such as trees [**6, 7, 13**], cubic graphs [**4, 15, 17**], regular graphs [**1, 3**], planar graphs [**14**], balanced complete multipartite graphs [**20**] and complete graphs [**6, 8, 9, 10, 19**]. The linear 2-arboricity, the linear 3-arboricity and a lower bound of linear k-arboricity of balanced complete bipartite graphs are obtained in [**9, 10, 11**], respectively. In [**21**], Xue and Zuo determined the linear $(n-1)$-arboricity of a balanced complete multipartite graph $K_{n(m)}$.

In the present paper, we obtain the linear $(n-1)$-arboricity of $K_n \square K_{n,n}$, the Hamming graph K_n^m, and C_{nt}^m for positive integers n, m, and t, where C_{nt}^m is the Cartesian product $C_{nt} \square C_{nt} \square \cdots \square C_{nt}$ of m cycles C_{nt}.

2. MAIN RESULTS

As preparation, we need the following lemmas.

Lemma 2.1. *If $G = G_1 \cup G_2 \cup \cdots \cup G_n$, then*

$$\mathrm{la}_k(G) \leq \mathrm{la}_k(G_1) + \mathrm{la}_k(G_2) + \cdots + \mathrm{la}_k(G_n).$$

Lemma 2.2. *If H is a subgraph of G, then $\mathrm{la}_k(G) \geq la_k(H)$.*

As for a lower bound on $\mathrm{la}_k(G)$, since any vertex in a linear k-forest has degree at most 2 and a linear k-forest in a graph G has at most $\left\lfloor \frac{k \cdot \mid V(G) \mid}{k+1} \right\rfloor$ edges, the following result is obvious.

Lemma 2.3. *For any graph G with maximum degree $\triangle(G)$, then*

$$\mathrm{la}_k(G) \geq \max\left\{ \left\lceil \frac{\Delta(G)}{2} \right\rceil, \left\lceil \frac{|E(G)|}{\left\lfloor \frac{k|V(G)|}{k+1} \right\rfloor} \right\rceil \right\}.$$

Assume that G and H are graphs. A spanning subgraph F of G is called an H-*factor* if each component of F is isomorphic to H. If G is expressible as an edge-disjoint union of H-*factors*, then this union is called an H-*factorization.*

Lemma 2.4. [**20**] *If a graph G has an H-factorization with t H-factors, then* $\mathrm{la}_k(G) \leq t \cdot \mathrm{la}_k(H)$.

According to the definition of the Cartesian product graph $G_1 \square G_2 \square \cdots \square G_m$, we can obtain a G_p-factor such that the two ends $(u_1, u_2, \ldots, u_m)$ and $(v_1, v_2, \ldots, v_m)$ of each edge of the G_p-factor satisfy $u_p v_p \in E(G_p)$ and $u_i = v_i$ for all other $i \neq p$, where $p \in [1, m]$. Hence we have the following result.

Lemma 2.5. *Let $G = G_1 \square G_2 \square \cdots \square G_m$. Then G can be decomposed into the edge-disjoint union of a G_1-factor, a G_2-factor, …, and a G_m-factor.*

Therefore we have

$$\mathrm{la}_k(G_1 \square G_2 \square \cdots \square G_n) \leq \mathrm{la}_k(G_1) + \mathrm{la}_k(G_2) + \cdots + \mathrm{la}_k(G_n).$$

Let $G = K_{n_1} \square K_{n_2} \square \cdots \square K_{n_m}$ be a Hamming graph, then G can be decomposed into the edge-disjoint union of a K_{n_1}-factor, a K_{n_2}-factor, …, and a K_{n_m}-factor, so K_n^m has a K_n-factorization that contains m K_n-factors. It is obvious that $G = C_{nt}^m$ has a C_{nt}-factorization which contains m C_{nt}-factors. Similarly, $K_n \square K_{n,n}$ can be decomposed into the edge-disjoint union of a K_n-factor and a $K_{n,n}$-factor.

2.1. The linear $(n-1)$-arboricity of Hamming graph K_n^m

Lemma 2.6. [9] *Let* $V(K_{2n}) = \{v_0, v_1, \ldots, v_{2n-1}\}$. *For* $i \in [0, n-1]$, *put*

$$P_i = v_{0+i}v_{1+i}v_{2n-1+i}v_{2+i}v_{2n-2+i}\cdots v_{n+1+i}v_{n+i}$$

where the subscripts of v_j *are taken modulo* $2n$. *Then* $P_i, i \in [0, n-1]$, *are disjoint Hamiltonian paths of the complete graph* K_{2n}.

Lemma 2.7. [9] *Let* $n = 2k+1 \geq 3$, *and* $V(K_n) = \{v_0, v_1, \ldots, v_{2k-1}, u\}$. *Then* K_n *can be decomposed into* k *edge-disjoint Hamilton cycles*

$$C_i = uv_{0+i}v_{1+i}v_{2k-1+i}v_{2+i}v_{2k-2+i}\cdots v_{k+1+i}v_{k+i}u$$

for $i \in [0, k-1]$, *where the subscripts of* v_j *are taken modulo* $2k$.

Lemma 2.8. [5] *For* $n \geq 3$, *the complete graph* K_n *is decomposable into edge-disjoint Hamilton cycles if and only if* n *is odd. For* $n \geq 2$, *the complete graph* K_n *is decomposable into edge-disjoint Hamilton paths if and only if* n *is even.*

Corollary 2.9. *The complete graph* K_n *can be decomposed into the edge-disjoint union of* $\frac{n-1}{2}$ *Hamilton paths and a path with length* $\frac{n-1}{2}$ *if* $n \geq 3$ *is odd.*

Proof. Let $n = 2k+1 \geq 3$, and $V(K_n) = \{v_0, v_1, \ldots, v_{2k-1}, u\}$. By Lemmas 2.7 and 2.8, K_n can be decomposed into k edge-disjoint Hamilton cycles

$$C_i = uv_{0+i}v_{1+i}v_{2k-1+i}v_{2+i}v_{2k-2+i}\cdots v_{k+1+i}v_{k+i}u$$

for $i \in [0, k-1]$, where subscripts of all v_j are taken modulo $2k$. Next, we take away one edge $v_{0+i}v_{1+i}$ from each Hamilton cycle C_i for $i \in [0, k-1]$, then C_i becomes a Hamilton path and all edges taken away form a path $v_0v_1\cdots v_k$ whose length is $k = \frac{n-1}{2}$.

Lemma 2.10. [2] $\mathrm{la}(K_n) = \mathrm{la}_{n-1}(K_n) = \left\lceil \frac{n}{2} \right\rceil$, *where* K_n *is a complete graph.*

Corollary 2.11. $G = K_n^m$ *has a* K_n^2*-factorization with* $\frac{m}{2}$ K_n^2*-factors if* m *is even, and* $G = K_n^m$ *can be decomposed into the edge-disjoint union of* $\frac{m-1}{2}$ K_n^2*-factors and a* K_n*-factor if* m *is odd.*

Proof. By Lemma 2.5, $G = K_n^m$ can be decomposed into the edge-disjoint union of m K_n-factors. It is not difficult to see that any two K_n-factors can form a K_n^2-factor. □

In the following, we discuss the linear $(n-1)$-arboricity of the Hamming graph K_n^m.

Theorem 2.12. $\mathrm{la}_{n-1}(K_n^2) = n$ *for every integer* $n \geq 3$.

Proof. By Lemma 2.5, K_n^2 has a K_n-factorization with 2 K_n-factors, denoted by G_1 and G_2. Every G_i, for $i \in [1,2]$, consists of n K_n.

If $n \geq 3$ is odd, then K_n can be decomposed into the edge-disjoint union of $\frac{n-1}{2}$ Hamilton paths and a path whose length is $\frac{n-1}{2}$ by Corollary 2.9.

Next, from each K_n in G_i for $i \in [1,2]$, we take away $\frac{n-1}{2}$ edges that induce a path of length $\frac{n-1}{2}$. Furthermore, all edges that taken away form a linear $(n-1)$-forest of G.

Let $K_n^2 = K_{n_1} \Box K_{n_2}$ (where $n_1 = n_2 = n$), $V(K_{n_1}) = \{x_p : p \in [0, n-1]\}$, and $V(K_{n_2}) = \{y_p : p \in [0, n-1]\}$. For the sake of convenience, we denote any vertex $(x_i, y_j) \in V(K_n^2)$ by i_j. For a fixed $i \in [0, n-1]$, all vertices in $\{i_p : p \in [0, n-1]\}$ induce a complete graph $K_n^{(i)}$. Now, for each $K_n^{(i)}$, we take away $\frac{n-1}{2}$ edges that induce a path

$$P_1^i = i_i i_{i+1} i_{i+2} \cdots i_{i+\frac{n-1}{2}},$$

where the subscripts of all i_j are taken modulo n. For a fixed $j \in [0, n-1]$, all vertices in $\{p_j : p \in [0, n-1]\}$ induce a complete graph $K_n^{(j)}$. For each $K_n^{(j)}$, we take away $\frac{n-1}{2}$ edges that induce a path

$$P_2^j = j_j (j+1)_j (j+2)_j \cdots \Big(j + \frac{n-1}{2}\Big)_j,$$

where the indices p of all p_j are taken modulo n. It is not difficult to see that $P_1^i \cup P_2^j$ is a path with length $n-1$ when $i = j$ and $V(P_1^i) \cap V(P_2^j) = \emptyset$ when $i \neq j$. So all edges taken away form a linear $(n-1)$-forest of G where the length of each path is $n-1$. All remaining edges form $n-1$ linear $(n-1)$-forests. Thus $\mathrm{la}_{n-1}(K_n^2) \leq n$ for odd n.

If n is even, then $\mathrm{la}_{n-1}(K_n) = n/2$ by Lemma 2.8, so we have $\mathrm{la}_{n-1}(K_n^2) \leq n$ by Lemma 2.5.

Since $\mathrm{la}_{n-1}(K_n^2) \geq n$ by Lemma 2.3, we have $\mathrm{la}_{n-1}(K_n^2) = n$.

Corollary 2.13. $\mathrm{la}(K_n^2) = n$ *for every positive integer* n.

Proof. Clearly, $\mathrm{la}(K_n^2) \leq \mathrm{la}_{n-1}(K_n^2) = n$ by Theorem 2.12. Since $|V(K_n^2)| = n^2$ and the largest number of edges in a linear forest is $n^2 - 1$, it is easy to verify that

$$\mathrm{la}(K_n^2) \geq \left\lceil \frac{|E(K_n^2)|}{n^2 - 1} \right\rceil = \left\lceil \frac{n^2(n-1)}{n^2-1} \right\rceil = n.$$

Hence we have $\mathrm{la}(K_n^2) = n$.

Theorem 2.14. $\mathrm{la}_{n-1}(K_n^m) = \lceil nm/2 \rceil$ *for all positive integers* m *and* $n \geq 2$.

Proof. Applying Lemma 2.3, we obtain that $\mathrm{la}_{n-1}(K_n^m) \geq \lceil mn/2 \rceil$. We will show the upper bound according to the parity of n.

Case 1. n is even.

By Lemmas 2.4, 2.5 and 2.10, $\mathrm{la}_{n-1}(K_n^m) \le m \cdot \mathrm{la}_{n-1}(K_n) = mn/2$.

Case 2. n is odd.
If m is even, then by Lemmas 2.4, 2.5 and Theorem 2.12, we obtain that

$$\mathrm{la}_{n-1}(K_n^m) \le \frac{m}{2}\, \mathrm{la}_{n-1}(K_n^2) = mn/2.$$

If m is odd, then by Lemmas 2.1, 2.4, 2.5 and Theorem 2.12, we have

$$\mathrm{la}_{n-1}(K_n^m) \le \frac{m-1}{2}\, \mathrm{la}_{n-1}(K_n^2) + \mathrm{la}_{n-1}(K_n) = (mn+1)/2 = \lceil mn/2 \rceil.$$

In a word, we have shown that

$$\mathrm{la}_{n-1}(K_n^m) = \lceil nm/2 \rceil$$

for all positive integers m and $n \ge 2$.

2.2. The linear $(n-1)$-arboricity of Cartesian product graph C_{nt}^m

First, we study the linear $(n-1)$-arboricity of Cartesian product graph C_n^m in the following.

Theorem 2.15. $\mathrm{la}_{n-1}(C_n^m) = m + \left\lceil \frac{m}{n-1} \right\rceil$ *for positive integers* m *and* $n \ge 3$.

Proof. By Lemma 2.3, it is not difficult to verify that

$$\mathrm{la}_{n-1}(C_n^m) \ge m + \left\lceil \frac{m}{n-1} \right\rceil.$$

We will obtain the upper bound after the following three claims are proved.

Claim 1. $\mathrm{la}_{n-1}(C_n^{n-1}) \le n$.
By Lemma 2.5, C_n^{n-1} can be decomposed into the edge-disjoint union of a C_{t_1}-factor, a C_{t_2}-factor, $\ldots$, and a $C_{t_{n-1}}$-factor, where $t_1 = t_2 = \cdots = t_{n-1} = n$. The C_{t_1}-factor contains n^{n-2} vertex-disjoint n-cycles, which are

$$\left(\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}, 1, 1, \ldots, 1\right), \left(\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}, 2, 1, \ldots, 1\right), \ldots, \left(\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}, n, n, \ldots, n\right),$$

where

$$\left(\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}, i, j, \ldots, k\right)$$

represents an n-cycle

$$(1, i, j, \cdots, k)(2, i, j, \cdots, k)(3, i, j, \cdots, k)\cdots(n, i, j, \cdots, k)(1, i, j, \cdots, k).$$

Similarly, the C_{t_2}-factor contains n^{n-2} vertex-disjoint n-cycles, which are

$$\left(1, \begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}, 1, 1, \dots, 1\right), \left(2, \begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}, 1, 1, \dots, 1\right), \dots, \left(n, \begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}, n, n, \dots, n\right),$$

where

$$\left(\ell, \begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}, i, j, \dots, k\right)$$

represents an n-cycle

$$(\ell, 1, i, j, \dots, k)(\ell, 2, i, j, \dots, k)(\ell, 3, i, j, \dots, k)\cdots(\ell, n, i, j, \dots, k)(\ell, 1, i, j, \dots, k),$$

$\dots$, and the $C_{t_{n-1}}$-factor contains n^{n-2} vertex-disjoint n-cycles, which are

$$\left(1, 1, \dots, 1, \begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}\right), \left(2, 1, \dots, 1, \begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}\right), \dots, \left(n, n, \dots, n, \begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}\right),$$

where

$$\left(i, j, \dots, k, \begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}\right)$$

represents an n-cycle

$$(i, j, \dots, k, 1)(i, j, \dots, k, 2)(i, j, \dots, k, 3)\cdots(i, j, \dots, k, n)(i, j, \dots, k, 1).$$

For the sake of convenience, we will denote the edge

$$(i, \dots, j, p, \dots, k)(i, \dots, j, p+1, \dots, k)$$

of C_{t_p}-factor by

$$\left(i, \dots, j, \begin{pmatrix}p\\p+1\end{pmatrix}, \dots, k\right).$$

Suppose that integers $\ell_1, \ell_2, \dots, \ell_{n-2} \in [1, n]$. Now, we take away one edge from every n-cycle in C_{t_u}-factor for $u \in [1, n-1]$: take away the edge

$$\left(\begin{pmatrix}\ell_1 - \sum_{k=2}^{n-2} \ell_k + 1\\ \ell_1 - \sum_{k=2}^{n-2} \ell_k\end{pmatrix}, \ell_1, \ell_2, \dots, \ell_{n-2}\right)$$

from

$$\left(\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix},\ell_1,\ell_2,\ldots,\ell_{n-2}\right),$$

take away

$$\left(\ell_1-\sum_{k=2}^{n-2}\ell_k,\binom{\ell_1}{\ell_1+1},\ell_2,\ldots,\ell_{n-2}\right)$$

from

$$(\ell_1-\sum_{k=2}^{n-2}\ell_k,\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix},\ell_2,\cdots,\ell_{n-2}),$$

take away

$$\left(\ell_1-\sum_{k=2}^{n-2}\ell_k,\ell_1+1,\binom{\ell_2}{\ell_2+1},\ell_3,\ldots,\ell_{n-2}\right)$$

from

$$\left(\ell_1-\sum_{k=2}^{n-2}\ell_k,\ell_1+1,\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix},\ell_3,\ldots,\ell_{n-2}\right),$$

$\cdots$, and take away

$$\left(\ell_1-\sum_{k=2}^{n-2}\ell_k,\ell_1+1,\ell_2+1,\ldots,\ell_{n-3}+1,\binom{\ell_{n-2}}{\ell_{n-2}+1}\right)$$

from

$$\left(\ell_1-\sum_{k=2}^{n-2}\ell_k,\ell_1+1,\ell_2+1,\ldots,\ell_{n-3}+1,\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}\right),$$

where all numbers are taken modulo n and belong to $[1,n]$.

It is not difficult to verify that all edges taken away can just form n^{n-2} vertex disjoint $n-1$ paths, which are

$$\left(\begin{pmatrix}\ell_1-\sum_{k=2}^{n-2}\ell_k+1\\ \ell_1-\sum_{k=2}^{n-2}\ell_k\end{pmatrix},\ell_1,\ell_2,\ldots,\ell_{n-2}\right)$$

$$\left(\ell_1-\sum_{k=2}^{n-2}\ell_k,\binom{\ell_1}{\ell_1+1},\ell_2,\ldots,l_{n-2}\right)$$

$$\left(\ell_1 - \sum_{k=2}^{n-2} \ell_k, \ell_1 + 1, \binom{\ell_2}{\ell_2 + 1}, \ell_3, \ldots, \ell_{n-2}\right) \cdots$$
$$\left(\ell_1 - \sum_{k=2}^{n-2} \ell_k, \ell_1 + 1, \ldots, \ell_{n-3} + 1, \binom{\ell_{n-2}}{\ell_{n-2} + 1}\right),$$

for $\ell_1, \ell_2, \ldots, \ell_{n-2} \in [1, n]$, respectively, where all numbers are taken modulo n and belong to $[1, n]$.

Hence all edges taken away can just form a linear $(n-1)$-forest, and all edges left in C_{n_i}-factor can just form a linear $(n-1)$-forest for each $i \in [1, n-1]$. Hence $\mathrm{la}_{n-1}(C_n^{n-1}) \leq n - 1 + 1 = n$.

Clearly, we can obtain the following Claim 2 as Claim 1 similarly.

Claim 2. $\mathrm{la}_{n-1}(C_n^r) \leq r + 1$, where $0 < r < n - 1$.

Now we show that the upper bound for the case of $m \geq n$.

Claim 3. $\mathrm{la}_{n-1}(C_n^m) \leq m + \left\lceil \frac{m}{n-1} \right\rceil$ for $m \geq n$.

Let $m = k(n-1)+r$. If $r \neq 0$, then C_n^m can be decomposed into k C_n^{n-1}-factors and a C_n^r-factor by Lemma 2.5, hence we have

$$\mathrm{la}_{n-1}(C_n^m) \leq k \cdot \mathrm{la}_{n-1}(C_n^{n-1}) + \mathrm{la}_{n-1}(C_n^r) = kn + r + 1 = m + \left\lceil \frac{m}{n-1} \right\rceil.$$

If $r = 0$, then C_n^m can be decomposed into k C_n^{n-1}-factors, so

$$\mathrm{la}_{n-1}(C_n^m) \leq k \cdot \mathrm{la}_{n-1}(C_n^{n-1}) = kn.$$

Therefore, we obtain that $\mathrm{la}_{n-1}(C_n^m) = m + \left\lceil \frac{m}{n-1} \right\rceil$ for positive integers m and $n \geq 3$. □

Secondly, we study the linear $(n-1)$-arboricity of C_{nt}^m.

Theorem 2.16. $\mathrm{la}_{n-1}(C_{nt}^m) = m + \left\lceil \frac{m}{n-1} \right\rceil$ *for all positive integers* m, $n \geq 3$ *and* $t \geq 1$.

Proof. The procedure is similarly as Theorem 2.15. For the sake of completeness, we give a sketch of the proof here. It is obvious that

$$\mathrm{la}_{n-1}(C_{nt}^m) \geq m + \left\lceil \frac{m}{n-1} \right\rceil$$

by Lemma 2.3.

Claim 1. $\mathrm{la}_{n-1}(C_{nt}^{n-1}) \leq n$.

Clearly, C_{nt}^{n-1} can be decomposed into the edge-disjoint union of a C_{nt_1}-factor, a C_{nt_2}-factor, ..., and a $C_{nt_{n-1}}$-factor, where $t_1 = t_2 = \cdots = t_{n-1} = t$.

Every factor contains $(nt)^{n-2}$ vertex-disjoint nt-cycles, which are

$$\left(\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix},1,1,\ldots,1\right),\left(\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix},2,1,\ldots,1\right),\ldots,\left(\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix},nt,nt,\ldots,nt\right);$$

$$\left(1,\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix},1,\ldots,1\right),\left(2,\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix},1,\ldots,1\right),\ldots,\left(nt,\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix},nt,\ldots,nt\right)$$

$$\vdots$$

and

$$\left(1,1,\ldots,1,\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix}\right),\left(2,1,\ldots,1,\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix}\right),\ldots,\left(nt,nt,\ldots,nt,\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix}\right),$$

respectively.

We take away t edges from every nt-cycle in C_{nt_q}-factor for $q \in [1, n-1]$: take away

$$\left(\begin{pmatrix}\ell_1-\sum_{u=2}^{n-2}\ell_u+1+pn\\ \ell_1-\sum_{u=2}^{n-2}\ell_u+pn\end{pmatrix},\ell_1,\ell_2,\ldots,\ell_{n-2}\right)$$

from

$$\left(\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix},\ell_1,\ell_2,\ldots,\ell_{n-2}\right),$$

take away

$$\left(\ell_1-\sum_{u=2}^{n-2}\ell_u+pn,\begin{pmatrix}\ell_1\\ \ell_1+1\end{pmatrix},\ell_2,\ldots,\ell_{n-2}\right)$$

from

$$\left(\ell_1-\sum_{u=2}^{n-2}\ell_u+pn,\begin{pmatrix}1\\2\\\vdots\\nt\end{pmatrix},\ell_2,\ldots,\ell_{n-2}\right),$$

take away

$$\left(\ell_1-\sum_{u=2}^{n-2}\ell_u+pn,\ell_1+1,\begin{pmatrix}\ell_2\\ \ell_2+1\end{pmatrix},\ell_3,\ldots,\ell_{n-2}\right)$$

from

$$\left(\ell_1 - \sum_{k=2}^{n-2} \ell_k + pn, \ell_1 + 1, \begin{pmatrix} 1 \\ 2 \\ \vdots \\ nt \end{pmatrix}, \ell_3, \dots, \ell_{n-2}\right),$$

$\cdots$, and take away

$$\left(\ell_1 - \sum_{u=2}^{n-2} \ell_u + pn, \ell_1 + 1, \ell_2 + 1, \dots, \ell_{n-3} + 1, \begin{pmatrix} \ell_{n-2} \\ \ell_{n-2} + 1 \end{pmatrix}\right)$$

from

$$\left(\ell_1 - \sum_{u=2}^{n-2} \ell_u + pn, \ell_1 + 1, \ell_2 + 1, \dots, \ell_{n-3} + 1, \begin{pmatrix} 1 \\ 2 \\ \vdots \\ nt \end{pmatrix}\right),$$

where $\ell_1, \ell_2, \dots, \ell_{n-2} \in [1, nt]$, $p \in [0, t-1]$, and all numbers are taken modulo nt and belong to $[1, nt]$.

It is not difficult to show that all edges taken away can just form $t(nt)^{n-2}$ vertex-disjoint $(n-1)$-paths, which are

$$\left(\begin{pmatrix} \ell_1 - \sum_{u=2}^{n-2} \ell_u + 1 + pn \\ \ell_1 - \sum_{u=2}^{n-2} \ell_u + pn \end{pmatrix}, \ell_1, \ell_2, \dots, \ell_{n-2}\right)$$

$$\left(\ell_1 - \sum_{u=2}^{n-2} \ell_u + pn, \begin{pmatrix} \ell_1 \\ \ell_1 + 1 \end{pmatrix}, \ell_2, \dots, \ell_{n-2}\right)$$

$$\left(\ell_1 - \sum_{u=2}^{n-2} \ell_u + pn, \ell_1 + 1, \begin{pmatrix} \ell_2 \\ \ell_2 + 1 \end{pmatrix}, \ell_3, \dots, \ell_{n-2}\right) \cdots$$

$$\left(\ell_1 - \sum_{u=2}^{n-2} \ell_u + pn, \ell_1 + 1, \dots, \ell_{n-3} + 1, \begin{pmatrix} \ell_{n-2} \\ \ell_{n-2} + 1 \end{pmatrix}\right),$$

respectively, where $p \in [0, t-1]$, $l_1, l_2, \dots, l_{n-2} \in [1, nt]$, and all numbers are taken modulo nt and belong to $[1, nt]$.

Thus all edges taken away can just form a linear $(n-1)$-forest, and all edges that are left in C_{nt_i}-factor can just form a linear $(n-1)$-forest for each $i \in [1, n-1]$, so $\mathrm{la}_{n-1}(C_{nt}^{n-1}) \le n - 1 + 1 = n$, and then Claim 1 holds.

Claim 2. $\mathrm{la}_{n-1}(C_{nt}^{r}) \le r + 1$, where $0 < r < n - 1$.

The proof is similar as Claim 1.

Claim 3. $\mathrm{la}_{n-1}(C_{nt}^{m}) \le m + \left\lceil \dfrac{m}{n-1} \right\rceil$ for $m \ge n$.

Let $m = d(n-1)+r$. If $r \neq 0$, then C_{nt}^m can be decomposed into d C_{nt}^{n-1}-factors and a C_{nt}^r-factor, so

$$\mathrm{la}_{n-1}(C_{nt}^m) \leq d \cdot \mathrm{la}_{n-1}(C_{nt}^{n-1}) + \mathrm{la}_{n-1}(C_{nt}^r) = dn + r + 1 = m + \left\lceil \frac{m}{n-1} \right\rceil.$$

If $r = 0$, then C_{nt}^m can be decomposed into d C_{nt}^{n-1}-factors, so

$$\mathrm{la}_{n-1}(C_{nt}^m) \leq d \cdot \mathrm{la}_{n-1}(C_{nt}^{n-1}) = dn,$$

and thus Claim 3 is proved.

Therefore, we have $\mathrm{la}_{n-1}(C_{nt}^m) = m + \left\lceil \frac{m}{n-1} \right\rceil$ for all positive integers m, $n \geq 3$ and $t \geq 1$.

2.3. The linear $(n-1)$-arboricity of Cartesian product graph $K_n \square K_{n,n}$

Finally, we study the linear $(n-1)$-arboricity of Cartesian product graph $K_n \square K_{n,n}$. The following lemma is needed for the proof of our main result.

Lemma 2.17. ([**9**]) $\mathrm{la}_k(K_{n,n}) = \left\lceil \frac{n}{2} \right\rceil + 1$, *if* $n-1 \leq k \leq 2n-2$.

Theorem 2.18. $\mathrm{la}_{n-1}(K_n \square K_{n,n}) = n+1$.

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$, and $V(K_{n,n}) = \{u_1, u_2, \ldots, u_{2n}\}$. Then

$$V(K_n \square K_{n,n}) = \{(v_i, u_j) | i \in [1,n], j \in [1,2n]\}.$$

Clearly, the vertex subset $\{(v_i, u_j) | j \in [1,2n]\}$ induces a balanced complete bipartite graph, which is denoted by $K_{n,n}^i$, for $i \in [1,n]$, and the vertex subset $\{(v_i, u_j) | i \in [1,n]\}$ induces a complete graph, which is denoted by K_n^j, for $j \in [1,2n]$. It is obvious that $K_n \square K_{n,n}$ can be decomposed into n disjoint balanced complete bipartite graphs $K_{n,n}^i$ for $i \in [1,n]$ and $2n$ disjoint complete graphs K_n^j for $j \in [1,2n]$.

Applying Lemma 2.3, we have $\mathrm{la}_{n-1}(K_n \square K_{n,n}) \geq n+1$. We will show that $\mathrm{la}_{n-1}(K_n \square K_{n,n}) \leq n+1$ in the following.

Case 1. n is even.

It is obvious that

$$\mathrm{la}_{n-1}(K_n \square K_{n,n}) \leq \mathrm{la}_{n-1}(K_n) + \mathrm{la}_{n-1}(K_{n,n}) \leq n/2 + (n/2+1) = n+1$$

by Lemma 2.17.

Case 2. n is odd.

Subcase 2.1. $n = 3$.

We show that $\mathrm{la}_2(K_3\Box K_{3,3}) \leq 4$ by direct construction. Let

$$\begin{aligned}F_1 = \{&(v_1,u_1)(v_3,u_1)(v_2,u_1),(v_1,u_4)(v_3,u_4)(v_2,u_4),(v_2,u_2)(v_1,u_2)(v_3,u_2),\\&(v_2,u_5)(v_1,u_5)(v_3,u_5),(v_1,u_3)(v_2,u_3)(v_3,u_3),(v_1,u_6)(v_2,u_6)(v_3,u_6)\},\\F_2 = \{&(v_1,u_6)(v_1,u_1)(v_1,u_5),(v_2,u_1)(v_2,u_5)(v_2,u_3),(v_3,u_6)(v_3,u_3)(v_3,u_5),\\&(v_1,u_3)(v_1,u_4)(v_1,u_2),(v_2,u_4)(v_2,u_2)(v_2,u_6),(v_3,u_1)(v_3,u_4)(v_3,u_2)\},\\F_3 = \{&(v_1,u_1)(v_2,u_1)(v_2,u_6),(v_2,u_2)(v_3,u_2)(v_3,u_4),(v_1,u_5)(v_1,u_3)(v_3,u_3),\\&(v_1,u_4)(v_2,u_4)(v_2,u_3),(v_2,u_5)(v_3,u_5)(v_3,u_1),(v_1,u_2)(v_1,u_6)(v_3,u_6)\},\end{aligned}$$

and

$$\begin{aligned}F_4 = \{&(v_1,u_1)(v_1,u_4),(v_1,u_2)(v_1,u_5),(v_1,u_3)(v_1,u_6),(v_2,u_1)(v_2,u_4),\\&(v_2,u_2)(v_2,u_5),(v_2,u_3)(v_2,u_6),(v_3,u_1)(v_3,u_4),(v_3,u_2)(v_3,u_5),\\&(v_3,u_3)(v_3,u_6)\}.\end{aligned}$$

Then it is obvious that each F_i is a linear 2-forest for $i \in [1,4]$, and thus the result holds.

Subcase 2.2. $n > 3$.

For odd $n \geq 5$, each K_n^i can be decomposed into $(n-1)/2$ Hamilton paths and a matching M_i of size $(n-1)/2$, where

$$M_i = \{(v_1,u_i)(v_2,u_i),(v_3,u_i)(v_4,u_i),\ldots,(v_{n-2},u_i)(v_{n-1},u_i)\},$$

$i = 1,2,\ldots,2n$.

In the following we first show the fact that the balanced complete bipartite graph $K_{n,n}$ can be decomposed into $(n-1)/2$ linear $(n-1)$-forests F_i and Q, where each F_i consists of two paths of length $n-1$ for $i \in [1,(n-1)/2]$, and Q is a vertex-disjoint union of $(n-1)/2$ cycles of length four and an isolated edge.

Let $X = \{x_1,\ldots,x_n\}$ and $Y = \{y_1,\ldots,y_n\}$ be two parts of $K_{n,n}$. Now, we color each edge x_iy_j of $K_{n,n}$ with the entry in the i-th row and the j-th column in Table 1. It is not difficult to verify that each induced subgraph by edges with the same color λ is a linear $(n-1)$-forest F_λ consisting of two paths of length $n-1$ for $\lambda \in [1,(n-1)/2]$, each induced subgraphs by edges with the same color a_i, for $i \in [1,(n-1)/2]$, is a cycle C_{a_i} of length four, where $V(C_{a_i}) \cap V(C_{a_j}) = \emptyset$ since a_i and a_j lie in different row and different column in Table 1 for $i \neq j$, and the induced subgraph by edges with color $a_{(n+1)/2}$ is the edge $x_{(n+1)/2}y_{(n+1)/2}$. Thus, for odd $n \geq 5$, we can decompose $K_{n,n}$ into $(n-1)/2$ linear $(n-1)$-forests F_i which consist of two paths of length $n-1$ for $1 \leq i \leq (n-1)/2$, and Q that is a vertex-disjoint union of $(n-1)/2$ cycles of length four and an isolated edge. Hence, for each $j \in [1,n]$, $K_{n,n}^j$ can be decomposed into $(n-1)/2$ $2P_n$ and $Q_j = (n-1)/2C_4 \cup P_2$.

Now all edges

$$\{(v_{2i-1},u_1)(v_{2i},u_1),(v_{2i-1},u_2)(v_{2i},u_2),\ldots,(v_{2i-1},u_{2n})(v_{2i},u_{2n})\},$$

Q_{2i-1}, and Q_{2i} form $(n-1)/2(K_2\Box C_4)$ and one C_4, where $i \in [1,(n-1)/2]$. Since each $K_2\square C_4$ can be decomposed into two $2P_4$ (for example, we have

$$\begin{aligned}K_2\square C_4 = &\{(v_1,u_1)(v_1,u_4)(v_2,u_4)(v_2,u_1),(v_1,u_3)(v_1,u_2)(v_2,u_2)(v_2,u_3)\}\\ &\cup\{(v_1,u_2)(v_1,u_1)(v_2,u_1)(v_2,u_2),(v_1,u_4)(v_1,u_3)(v_2,u_3)(v_2,u_4)\}\end{aligned}$$

a_1	1	1	2	2	$\cdots$	$\cdots$	$\cdots$	$\frac{n-5}{2}$	$\frac{n-3}{2}$	$\frac{n-3}{2}$	$\frac{n-1}{2}$	a_1
$\frac{n-1}{2}$	a_2	1	1	2	$\cdots$	$\cdots$	$\cdots$	$\frac{n-5}{2}$	$\frac{n-5}{2}$	$\frac{n-3}{2}$	a_2	$\frac{n-1}{2}$
$\frac{n-1}{2}$	$\frac{n-1}{2}$	a_3	1	1	$\cdots$	$\cdots$	$\cdots$	$\frac{n-7}{2}$	$\frac{n-5}{2}$	a_3	$\frac{n-3}{2}$	$\frac{n-3}{2}$
$\frac{n-3}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	a_4	1	$\cdots$	$\cdots$	$\cdots$	$\frac{n-7}{2}$	a_4	$\frac{n-5}{2}$	$\frac{n-5}{2}$	$\frac{n-3}{2}$
$\frac{n-3}{2}$	$\frac{n-3}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	a_5	$\cdots$	$\cdots$	$\cdots$	a_5	$\frac{n-7}{2}$	$\frac{n-7}{2}$	$\frac{n-5}{2}$	$\frac{n-5}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$a_{\frac{n+1}{2}}$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	3	4	4	a_5	$\cdots$	$\cdots$	$\cdots$	a_5	1	1	2	2
2	3	3	a_4	4	$\cdots$	$\cdots$	$\cdots$	$\frac{n-1}{2}$	a_4	1	1	2
2	2	a_3	3	4	$\cdots$	$\cdots$	$\cdots$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	a_3	1	1
1	a_2	2	3	3	$\cdots$	$\cdots$	$\cdots$	$\frac{n-3}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	a_2	1
a_1	1	2	2	3	$\cdots$	$\cdots$	$\cdots$	$\frac{n-3}{2}$	$\frac{n-3}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	a_1

Table 1.

where $V(K_2)=\{v_1,v_2\}$ and $V(C_4)=\{u_1,u_2,u_3,u_4\}$) and $C_4=2P_3$, we have two isomorphic edge-disjoint linear 3-forests $(n-1)/2(2P_4)\cup P_3$. Clearly,

$$Q_n=(n-1)/2C_4\cup P_2=((n-1)/2P_3\cup P_2)\cup(n-1)/2P_3.$$

Thus, we can use two colors to color these edges.

Hence $\mathrm{la}_{n-1}(K_n\square K_{n,n})\le(n-1)/2+(n-1)/2+2=n+1$ for odd $n>3$.

Therefore, we have obtained that $\mathrm{la}_{n-1}(K_n\square K_{n,n})=n+1$. □

According to Theorems 2.14, 2.16, and 2.18, it is clear that Conjecture 1.1 holds for Hamming graph and Cartesian product graphs C_{nt}^m and $K_n\square K_{n,n}$.

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