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ANALYSIS OF DISCRETE FRACTIONAL OPERATORS

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In this paper, we introduce two new monotonicity concepts for a nonnegative or nonpositive valued function defined on a discrete domain. We give examples to illustrate connections between these new monotonicity concepts and the traditional ones. We then prove some monotonicity criteria based on the sign of the fractional difference operator of a function f, $\Delta^{\nu} f$ with $0 < \nu < 1$. As an application, we state and prove the mean value theorem on discrete fractional calculus.

1. INTRODUCTION

Despite the long and rich history of fractional calculus, discrete fractional calculus attracted mathematicians and scientists into its fairly new research area in a short period of time. In this time period, the theory of discrete fractional calculus has been developed in many directions parallel to the theory of fractional calculus such as fractional difference equations, discrete Mittag-Leffler functions, inequalities with discrete fractional operators, see [**1–18**] and the references therein.

While the mathematicians have been working on the theory and applications of fractional calculus, they faced the lack of not only having clear geometric meaning but also basic theory on the analysis of the fractional derivative and integral operators, such as mean value theorem. Most recently, monotonicity and convexity results obtained by Dahal and Goodrich in [**8**] and Goodrich in [**15**] initiated the development of analysis of the discrete fractional operators in the theory of discrete fractional calculus.

In the papers [**8, 15**], the authors obtained the following monotonicity and convexity results based on the sign of the fractional difference operator of a nonnegative real valued function defined on $\mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

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Theorem 1.1 ([**8**]). *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a nonnegative function satisfying $y(0) = 0$. Fix $\nu \in (1, 2)$ and suppose that $\Delta_0^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{2-\nu}$. Then y is increasing on $\mathbb{N}_0$.*

Theorem 1.2 ([**15**]). *Fix $\mu \in (N-1, N)$, for $N \in \mathbb{N}_3$ given, and let $y : \mathbb{N}_0 \to \mathbb{R}$ be a given function satisfying $\Delta^j y(0) = 0$ for each $j \in \{0, 1, 2, \ldots, N-3\}$, $\Delta^{N-2} y(0) \geq 0$, and $\Delta_0^\mu y(t) \geq 0$ for each $t \in \mathbb{N}_{N-\mu}$. Then $\Delta^{N-1} y(t) \geq 0$, for each $t \in \mathbb{N}_0$.*

We note that the above two results do not include the case where ν is between zero and one.

The main purpose of this paper is to obtain monotonicity results for $\nu \in (0, 1)$. First we introduce ν-increasing and ν-decreasing functions for any positive real number ν. We give some restrictions on ν to compare these new monotonicity concepts with the traditional ones. We restate the following monotonicity criterion of the discrete calculus in the discrete fractional calculus:

Let $f : \mathbb{N}_0 \to \mathbb{R}$.

f is monotone increasing on $\mathbb{N}_0$ if and only if $\Delta f(t) \geq 0$ for all $t \in \mathbb{N}_0$.

For this purpose we consider a forward fractional difference operator of Riemann-Liouville type as in the papers [**2–7**], We then prove some monotonicity criteria for a function f which is defined on $\mathbb{N}_0$ and has a sign (positive or negative) for $\Delta^\nu f$ when ν is between 0 and 1. As an application of our main result, we shall state and prove a mean value theorem in the discrete fractional calculus.

2. PRELIMINARIES

In this section we first present sufficient fundamental definitions and formulas so that the article is self-contained.

Let Γ denote the usual special Gamma function and recall the notation

$$t^{(\mu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}.$$

Throughout, we assume that if $t + 1 - \mu \in \{0, -1, \ldots, -k, \ldots\}$, then $t^{(\mu)} = 0$. We consider the forward fractional sum as defined by Miller and Ross [**16**]

$$\Delta_a^{-\nu} f(t) = \sum_{s=a}^{t-\nu} \frac{\big(t - \sigma(s)\big)^{(\nu-1)}}{\Gamma(\nu)} f(s),$$

where $\nu \geq 0$, $a \in \mathbb{R}$, and $\sigma(s) = s + 1$. Define $\mathbb{N}_{t_0} = \{t_0, t_0 + 1, t_0 + 2, \ldots\}$ and note that $\Delta_a^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$. Further, we shall consider the Riemann-Liouville fractional difference

$$\Delta_a^\mu f(t) = \Delta_a^{m-\nu} f(t) = \Delta^m (\Delta_a^{-\nu} f(t)),$$

where $\mu > 0$, $m - 1 < \mu \leq m$, m denotes a positive integer, and $-\nu = \mu - m$.

We recall the following power rule

$$\Delta t^{(\mu)} = \mu \Delta t^{(\mu-1)}. \tag{2.1}$$

Next we introduce two new monotonicity concepts. Let ν be any positive real number.

Definition 2.3. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \geq 0$. y is called a ν-increasing function on $\mathbb{N}_0$, if*

$$y(a+1) \geq \nu y(a) \text{ for all } a \in \mathbb{N}_0.$$

Note that if y is increasing on $\mathbb{N}_0$ and $0 < \nu < 1$, then y is ν-increasing on $\mathbb{N}_0$. Also, if y is ν-increasing on $\mathbb{N}_0$ and $\nu \geq 1$, then y is increasing on $\mathbb{N}_0$. If $\nu = 1$, then y is increasing on $\mathbb{N}_0$ if and only if y is ν-increasing on $\mathbb{N}_0$.

Definition 2.4. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \leq 0$. y is called a ν-decreasing function on $\mathbb{N}_0$, if*

$$y(a+1) \leq \nu y(a) \text{ for all } a \in \mathbb{N}_0.$$

Note that if y is decreasing on $\mathbb{N}_0$ and $0 < \nu < 1$, then y is ν-decreasing on $\mathbb{N}_0$. Also, if y is ν-decreasing on $\mathbb{N}_0$ and $\nu \geq 1$, then y is decreasing on $\mathbb{N}_0$. If $\nu = 1$, then y is decreasing on $\mathbb{N}_0$ if and only if y is ν-decreasing on $\mathbb{N}_0$.

EXAMPLE 2.1. Consider $g(t) = e^{-t}$ on $\mathbb{N}_0$. We claim that the function g is ν-increasing when $\nu \in (0, 1/e]$. This can be easily verified. In fact, we multiply each side of the inequality $0 < \nu \leq 1/e$ by e^{-t}. This implies that $0 < \nu e^{-t} \leq e^{-(1+t)}$. Therefore, by Definition 2.1, $g(t) = e^{-t}$ is ν-increasing on $\mathbb{N}_0$.

3. MAIN RESULTS

Theorem 3.5. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \geq 0$. Fix $\nu \in (0,1)$ and suppose that*

$$\Delta_0^{\nu} y(t) \geq 0 \text{ for each } t \in \mathbb{N}_{1-\nu}.$$

Then, y is ν-increasing on $\mathbb{N}_0$.

Proof. We will prove that y is ν-increasing by mathematical induction. First, we observe that

$$\Delta_0^{\nu} y(t) = \Delta \Delta_0^{-(1-\nu)} y(t) = \Delta \left[\frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} (t - \sigma(s))^{(-\nu)} y(s) \right] \geq 0.$$

Let $s(t) = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} (t - \sigma(s))^{(-\nu)} y(s)$. Since $\Delta s(t) \geq 0$, $s(t)$ is an increasing function on $\mathbb{N}_{1-\nu}$. This implies that

$$\begin{aligned} s(2-\nu)-s(1-\nu) &= \frac{1}{\Gamma(1-\nu)}(1-\nu)^{(-\nu)}y(0)+\frac{1}{\Gamma(1-\nu)}(-\nu)^{(-\nu)}y(1) \\ &\quad -\frac{1}{\Gamma(1-\nu)}(-\nu)^{(-\nu)}y(0) \\ &= \frac{1}{\Gamma(1-\nu)}\left[\Delta(-\nu)^{(-\nu)}y(0)+(-\nu)^{(-\nu)}y(1)\right] \\ &= \frac{1}{\Gamma(1-\nu)}\left[-\nu\frac{\Gamma(1-\nu)}{\Gamma(2)}y(0)+\frac{\Gamma(1-\nu)}{\Gamma(1)}y(1)\right] \geq 0. \end{aligned}$$

Therefore, we have

$$y(1) \geq \nu y(0).$$

Now, let us assume that the induction hypothesis is valid up to $n = k-1$. Hence we have

$$y(k) \geq \nu y(k-1) \geq \nu^2 y(k-2) \geq \cdots \geq \nu^k y(0) \geq 0. \tag{3.2}$$

We want to prove that for $n = k$, the inequality

$$y(k+1) \geq \nu y(k) \tag{3.3}$$

is valid. To prove (3.3) we first calculate,

$$s(k+1-\nu) = \frac{1}{\Gamma(1-\nu)}\sum_{s=0}^{k}\big(k+1-\nu-\sigma(s)\big)^{(-\nu)}y(s)$$

and

$$s(k+2-\nu) = \frac{1}{\Gamma(1-\nu)}\sum_{s=0}^{k+1}\big(k+2-\nu-\sigma(s)\big)^{(-\nu)}y(s).$$

Since $s(t)$ is increasing, we have

$$\begin{aligned} & s(k+2-\nu)-s(k+1-\nu) \\ &= \frac{1}{\Gamma(1-\nu)}\sum_{s=0}^{k+1}(k+1-\nu-s)^{(-\nu)}y(s)-\frac{1}{\Gamma(1-\nu)}\sum_{s=0}^{k}(k-\nu-s)^{(-\nu)}y(s) \geq 0. \end{aligned}$$

Performing the sum operations above, we have

$$\begin{aligned} & \frac{1}{\Gamma(1-\nu)}\big[(k+1-\nu)^{(-\nu)}y(0)+(k-\nu)^{(-\nu)}y(1) \\ &\quad +(k-1-\nu)^{(-\nu)}y(2)+\cdots+(2-\nu)^{(-\nu)}y(k-1) \\ &\quad +(1-\nu)^{(-\nu)}y(k)+(-\nu)^{(-\nu)}y(k+1)\big] \\ &\quad -\frac{1}{\Gamma(1-\nu)}\big[(k-\nu)^{(-\nu)}y(0)+(k-1-\nu)^{(-\nu)}y(1)+(k-2-\nu)^{(-\nu)}y(2) \\ &\quad +\cdots+(1-\nu)^{(-\nu)}y(k-1)+(-\nu)^{(-\nu)}y(k)\big] \geq 0. \end{aligned}$$

Then grouping the alike terms we obtain the following inequality:

$$\frac{1}{\Gamma(1-\nu)}\Big[[(k+1-\nu)^{(-\nu)}-(k-\nu)^{(-\nu)}]y(0)+\big[(k-\nu)^{(-\nu)}-(k-1-\nu)^{(-\nu)}\big]y(1)$$
$$+\big[(k-1-\nu)^{(-\nu)}-(k-2-\nu)^{(-\nu)}\big]y(2)+\cdots+\big[(2-\nu)^{(-\nu)}-(1-\nu)^{(-\nu)}\big]y(k-1)$$
$$+\big[(1-\nu)^{(-\nu)}-(-\nu)^{(-\nu)}\big]y(k)+(-\nu)^{(-\nu)}y(k+1)\Big]\geq 0.$$

Next, we rewrite the coefficients of $y(0), y(1), \ldots y(k+1)$ with Δ operator as follows

$$\frac{1}{\Gamma(1-\nu)}\big[\Delta(k-\nu)^{(-\nu)}y(0)+\Delta(k-1-\nu)^{(-\nu)}y(1)+\Delta(k-2-\nu)^{(-\nu)}y(2)$$
$$+\cdots+\Delta(1-\nu)^{(-\nu)}y(k-1)+\Delta(-\nu)^{(-\nu)}y(k)+(-\nu)^{(-\nu)}y(k+1)\big]\geq 0.$$

After applying the Δ operator using the power rule (2.1), we obtain

$$\frac{1}{\Gamma(1-\nu)}\big[(-\nu)(k-\nu)^{(-\nu-1)}y(0)+(-\nu)(k-1-\nu)^{(-\nu-1)}y(1)$$
$$+(-\nu)(k-2-\nu)^{(-\nu-1)}y(2)+\cdots+(-\nu)(1-\nu)^{(-\nu-1)}y(k-1)$$
$$+(-\nu)(-\nu)^{(-\nu-1)}y(k)+(-\nu)^{(-\nu)}y(k+1)\big]\geq 0.$$

Then using the definition of falling factorial power, we have

$$\frac{1}{\Gamma(1-\nu)}\Bigg[(-\nu)\frac{\Gamma(k-\nu+1)}{\Gamma(k-\nu+1+\nu+1)}y(0)+(-\nu)\frac{\Gamma(k-1-\nu+1)}{\Gamma(k-1-\nu+1+\nu+1)}y(1)$$
$$+(-\nu)\frac{\Gamma(k-2-\nu+1)}{\Gamma(k-2-\nu+1+\nu+1)}y(2)+\cdots+(-\nu)\frac{\Gamma(1-\nu+1)}{\Gamma(1-\nu+1+\nu+1)}y(k-1)$$
$$+(-\nu)\frac{\Gamma(-\nu+1)}{\Gamma(-\nu+1+\nu+1)}y(k)+\Gamma(-\nu+1)y(k+1)\Bigg]\geq 0.$$

Next we simplify the above expression as the following

$$y(k+1)+\frac{(-\nu)}{\Gamma(1-\nu)}\Bigg[\frac{\Gamma(k-\nu+1)}{\Gamma(k+2)}y(0)+\frac{\Gamma(k-\nu)}{\Gamma(k+1)}y(1)+\frac{\Gamma(k-1-\nu)}{\Gamma(k)}y(2)$$
$$+\cdots+\frac{\Gamma(2-\nu)}{\Gamma(3)}y(k-1)+\frac{\Gamma(1-\nu)}{\Gamma(2)}y(k)\Bigg]\geq 0.$$

So, we have

$$y(k+1)+\frac{(-\nu)}{\Gamma(1-\nu)}\Bigg[\frac{(k-\nu)(k-1-\nu)\cdots(2-\nu)(1-\nu)\Gamma(1-\nu)}{\Gamma(k+2)}y(0)$$
$$+\frac{(k-1-\nu)(k-2-\nu)\cdots(2-\nu)(1-\nu)\Gamma(1-\nu)}{\Gamma(k+1)}y(1)$$
$$+\frac{(k-2-\nu)(k-3-\nu)\cdots(2-\nu)(1-\nu)\Gamma(1-\nu)}{\Gamma(k)}y(2)$$
$$+\cdots+\frac{(1-\nu)\Gamma(1-\nu)}{\Gamma(3)}y(k-1)+\frac{\Gamma(1-\nu)}{\Gamma(2)}y(k)\Bigg]\geq 0.$$

Only by one simple algebraic step we obtain

$$\begin{aligned} y(k+1) \geq & \frac{(k-\nu)(k-1-\nu)\cdots(2-\nu)(1-\nu)\nu}{\Gamma(k+2)} y(0) \\ & + \frac{(k-1-\nu)(k-2-\nu)\cdots(2-\nu)(1-\nu)\nu}{\Gamma(k+1)} y(1) \\ & + \frac{(k-2-\nu)(k-3-\nu)\cdots(2-\nu)(1-\nu)\nu}{\Gamma(k)} y(2) \\ & + \cdots + \frac{(1-\nu)\nu}{\Gamma(3)} y(k-1) + \frac{\nu}{\Gamma(2)} y(k). \end{aligned}$$

By the induction assumption (3.2), we have

$$\begin{aligned} y(k+1) - \nu y(k) \geq & \frac{(k-\nu)(k-1-\nu)\cdots(2-\nu)(1-\nu)\nu}{\Gamma(k+2)} y(0) \\ & + \frac{(k-1-\nu)(k-2-\nu)\cdots(2-\nu)(1-\nu)\nu}{\Gamma(k+1)} y(1) \\ & + \frac{(k-2-\nu)(k-3-\nu)\cdots(2-\nu)(1-\nu)\nu}{\Gamma(k)} y(2) + \cdots + \frac{(1-\nu)\nu}{\Gamma(3)} y(k-1) \geq 0, \end{aligned}$$

Hence, we conclude that for each $k \in \mathbb{N}$,

$$y(k+1) - \nu y(k) \geq 0. \qquad \square$$

In the proof of the next theorem, the rising factorial power function plays an important role. For the convenience of the reader, we recall its definition (see [**1, 2**]). The rising factorial power function is defined by

$$t^{\overline{\nu}} = \frac{\Gamma(t+\nu)}{\Gamma(t)}.$$

We note that the Gamma function is not defined at zero and negative integers. Therefore we consider a map $t \to t^{\overline{\nu}}$ from the set $\{t \in \mathbb{R} : t \text{ and } t+\nu \text{ do not belong to } \mathbb{Z}^- \cup \{0\}\}$ to the set of real numbers $\mathbb{R}$.

Theorem 3.6. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \geq 0$. Fix $\nu \in (0,1)$ and assume that y is an increasing function on $\mathbb{N}_0$. Then,*

$$\Delta_0^\nu y(t) \geq 0 \text{ for each } t \in \mathbb{N}_{1-\nu}.$$

Proof. We want to show that

$$\Delta_0^\nu y(t) = \Delta \Delta_0^{-(1-\nu)} y(t) = \Delta \left[\frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} (t-\sigma(s))^{(-\nu)} y(s) \right] \geq 0.$$

Similarly, let

$$s(t) = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} (t-\sigma(s))^{(-\nu)} y(s).$$

To complete the proof, we need to show that $s(t)$ is increasing on $\mathbb{N}_{1-\nu}$.
For any natural number k with $k \geq 1$ we show that

$$s(k+1-\nu) - s(k-\nu) \geq 0,$$

is valid. In fact, we have

$$\begin{aligned}
&s(k+1-\nu) - s(k-\nu) \\
&= \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k} (k+1-\nu-\sigma(s))^{(-\nu)} y(s) - \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-1} (k-\nu-\sigma(s))^{(-\nu)} y(s) \\
&= \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-1} \Delta_k (k-\nu-\sigma(s))^{(-\nu)} y(s) + \frac{1}{\Gamma(1-\nu)} (-\nu)^{(-\nu)} y(k) \\
&= \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-1} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)} y(s) + y(k) \\
&= y(k) - \nu y(k-1) + \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-2} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)} y(s) \\
&= y(k) - \nu y(k-1) + \frac{\nu}{\Gamma(1-\nu)} \sum_{s=0}^{k-2} (k-\nu-\sigma(s))^{(-\nu-1)} (y(k-1) - y(s)) \\
&\quad + \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-2} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)} y(k-1) \\
&\geq y(k) - \nu y(k-1) + \frac{y(k-1)}{\Gamma(1-\nu)} \sum_{s=0}^{k-2} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)} \\
&= y(k) - y(k-1) + y(k-1) + \frac{y(k-1)}{\Gamma(1-\nu)} \sum_{s=0}^{k-1} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)} \\
&\geq y(k-1)\left(1 + \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-1} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)}\right) \\
&= y(k-1) \sum_{s=0}^{k} \frac{(-\nu)^{\overline{s}}}{s!} = y(k-1) \frac{(1-\nu)^{\overline{k}}}{k!} \ \geq 0,
\end{aligned}$$

□

The above proof can be easily carried over to the proof of the following result.

Theorem 3.7. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) > 0$. Fix $\nu \in (0,1)$ and assume that y is a strictly increasing function on $\mathbb{N}_0$. Then,*

$$\Delta_0^\nu y(t) > 0 \text{ for each } t \in \mathbb{N}_{1-\nu}.$$

Corollary 3.1. *Let $h : [1, +\infty)_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ be a nonnegative, continuous function, and let A be a nonnegative real number. Then the unique solution of the discrete* *fractional IVP*

$$\Delta_0^\nu y(t) = h(t+\nu-1, y(t+\nu-1)),\ t \in \mathbb{N}_{2-\nu}$$

$$y(0) = A,$$

y is ν-increasing and nonnegative.

In a similar way, the above results can be obtained for the function which takes negative value at the initial point of its domain.

Theorem 3.8. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \leq 0$. Fix $\nu \in (0,1)$ and suppose that*

$$\Delta_0^\nu y(t) \leq 0 \text{ for each } t \in \mathbb{N}_{1-\nu}.$$

Then, y is ν-decreasing on $\mathbb{N}_0$.

Theorem 3.9. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \leq 0$. Fix $\nu \in (0,1)$ and assume that y is decreasing function on $\mathbb{N}_0$. Then,*

$$\Delta_0^\nu y(t) \leq 0 \text{ for each } t \in \mathbb{N}_{1-\nu}.$$

Corollary 3.2. *Let $h : [1,+\infty)_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ be a nonpositive, continuous function, and let A be a nonpositive real number. Then the unique solution of the discrete fractional IVP*

$$\Delta_0^\nu y(t) = h(t+\nu-1, y(t+\nu-1)),\ t \in \mathbb{N}_{2-\nu}, \qquad y(0) = A,$$

y is ν-decreasing and nonpositive.

4. AN APPLICATION

The fractional difference operator $\Delta_c^{-\nu}$ maps functions defined on $\mathbb{N}_c$ to functions defined on $\mathbb{N}_{c+\nu}$. Hence the following equality is defined for all $t \in \mathbb{N}_{c+\nu}$

$$\Delta_c^{-\nu} f(t) = \sum_{s=c}^{t-\nu} \frac{(t-\sigma(s))^{(\nu-1)}}{\Gamma(\nu)} f(s),$$

where c is any real number. If $a \in \mathbb{N}_c$, then it makes sense to calculate $\Delta_a^{-\nu} f(t)$ for all $t \in \mathbb{N}_{a+\nu}$. Such a characteristic property of this operator leads us to obtain the fundamental theorem of discrete fractional calculus.

Theorem 4.10. *Let f be defined on $\mathbb{N}_c$ and $a, b \in \mathbb{N}_c$ with $a < b$. Then the following equality holds:*

$$\Delta_{a-\nu+1}^{-\nu} \Delta_a^\nu f(t)|_{t=b} = f(b) - \frac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} f(a),$$

where $\nu \in (0,1)$.

To prove the above equality, we use Theorem 2.1 in [**3**] which states that for any $\nu > 0$ the following equality holds:

$$\Delta_c^{-\nu}\Delta f(t) = \Delta\Delta_c^{-\nu} f(t) - \frac{(t-c)^{(\nu-1)}}{\Gamma(\nu)} f(c),$$

where f is defined on $\mathbb{N}_c$.
Then we also use Theorem 2.2 in [**5**] which states that for all $\nu, \mu > 0$,

$$\Delta_\nu^{-\mu}[\Delta_0^{-\nu} f(t)] = \Delta_0^{-(\nu+\mu)} f(t) = \Delta_\mu^{-\nu}[\Delta_0^{-\mu} f(t)],$$

for all t such that $t \equiv (\mu + \nu) \mod (1)$.

Proof. We start the proof with manipulating the left side of the equality using Theorem 2.1 in [**3**].

$$\begin{aligned}
&\Delta_{a-\nu+1}^{-\nu}\Delta_a^\nu f(t)\big|_{t=b}\\
&= \Delta\Delta_{a-\nu+1}^{-\nu}\Delta_a^{-(1-\nu)} f(t)\big|_{t=b} - \frac{(t-(a-\nu+1))^{(\nu-1)}}{\Gamma(\nu)}\Big|_{t=b}\Delta_a^{-(1-\nu)} f\big|_{t=a-\nu+1}.\\
&= \Delta\Delta_a^{-1} f(t)\big|_{t=b} - \frac{(b-(a-\nu+1))^{(\nu-1)}}{\Gamma(\nu)} f(a) = f(b) - \frac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} f(a),
\end{aligned}$$

where we used Theorem 2.2 in [**5**] and the identity $\Delta_a^{-(1-\nu)} f\big|_{t=a-\nu+1} = f(a)$. □

Now we are in a position to state and to prove the mean value theorem of discrete fractional calculus.

Theorem 4.11. *Let f and g be defined on $\mathbb{N}_c$ and g be strictly increasing function on $[a,b] \cap \mathbb{N}_c$ and satisfying $g(a) > 0$, where $a, b \in \mathbb{N}_c$ with $a < b$. Then there exist $\tau_1, \tau_2 \in [a,b]$ such that*

$$\frac{\Delta_a^\nu f(\tau_1)}{\Delta_a^\nu g(\tau_1)} \le \frac{f(b) - \dfrac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} f(a)}{g(b) - \dfrac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} g(a)} \le \frac{\Delta_a^\nu f(\tau_2)}{\Delta_a^\nu g(\tau_2)},$$

where $\nu \in (0,1)$.

Proof. Suppose to the contrary that

$$\frac{f(b) - \dfrac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} f(a)}{g(b) - \dfrac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} g(a)} > \frac{\Delta_a^\nu f(t)}{\Delta_a^\nu g(t)} \tag{4.4}$$

or

$$\frac{f(b) - \dfrac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} f(a)}{g(b) - \dfrac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} g(a)} < \frac{\Delta_a^\nu f(t)}{\Delta_a^\nu g(t)} \tag{4.5}$$

for all $t \in [a, b]$. Since g is strictly increasing, we use Theorem 3.3 to conclude that $\Delta_a^\nu g(t) > 0$ on the discrete interval $[a+1-\nu, b+1-\nu]$. Hence the above inequality (4.4) becomes

$$\frac{f(b) - \dfrac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} f(a)}{g(b) - \dfrac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} g(a)} \Delta_a^\nu g(t) > \Delta_a^\nu f(t).$$

Since the fractional sum operator is inequality preserving and linear, we apply $\Delta_{a-\nu+1}^{-\nu}$ at $t = b$ to each side of the above inequality to obtain

$$\frac{f(b) - \dfrac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} f(a)}{g(b) - \dfrac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} g(a)} \Delta_{a-\nu+1}^{-\nu} \Delta_a^\nu g(t)\big|_{t=b} > \Delta_{a-\nu+1}^{-\nu} \Delta_a^\nu f(t)\big|_{t=b}.$$

As a result of Theorem 4.10, we obtain the following

$$f(b) - \frac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} f(a) > f(b) - \frac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} f(a)$$

which is a contradiction. In a similar way, one can see that the inequality (4.5) leads a contradiction. This completes the proof.

Remark 4.1. In the statement of the Theorem 4.11, the quantity

$$g(b) - \frac{(b-a+\nu-1)^{(\nu-1)}}{\Gamma(\nu)} g(a)$$

is not equal to zero. Indeed, if we assume that it is equal to zero, then we obtain

$$\frac{g(b)}{g(a)} = \frac{\Gamma(b-a+\nu)}{\Gamma(\nu)\Gamma(b-a+1)} < 1.$$

This implies that $g(b) < g(a)$. This contradicts with the hypothesis of the theorem.

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