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EXISTENCE OF SOLUTIONS FOR HYBRID FRACTIONAL PANTOGRAPH EQUATIONS

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In this paper, we study the existence of the hybrid fractional pantograph equation

$$\begin{cases} D_{0+}^{\alpha}\Big[\dfrac{x(t)}{f(t,x(t),x(\mu t))}\Big] = g\big(t,x(t),x(\sigma t)\big),\ 0<t<1, \\ x(0)=0, \end{cases}$$

where $\alpha,\mu,\sigma \in (0,1)$ and D_{0+}^{α} denotes the Riemann-Liouville fractional derivative. The results are obtained using the technique of measures of noncompactness in the Banach algebras and a fixed point theorem for the product of two operators verifying a Darbo type condition. Some examples are provided to illustrate our results.

1. INTRODUCTION

Fractional differential equations are a very important tool in modelling many phenomena of physics and, therefore, they deserve an independent study of their theories parallel to the well-known theory of differential equations, [**10, 12, 15, 17**]. On the other hand, a great number of papers about differential and integral equations with a modified argument have appeared in the literature recently. Such equations arise in a wide variety of applications such as the modelling of problems from the natural and social sciences, for example, physics, biology and economics. A special class of these equations is the differential equation with affine modification of the argument which can be delay differential equations or differential equations

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with linear modification of the argument. Results concerning with such kind of equations appear in the papers [**5, 6, 7, 8, 11, 14, 16, 18, 20**], for example.

One of the very special differential equations with linear modification of the argument is the pantograph equation

$$\text{(1)} \qquad \begin{cases} y'(t) = ay(t) + by(\lambda t),\ 0 \le t \le T, \\ y(0) = y_0, \end{cases}$$

where $0 < \lambda < 1$. This equation appears in different fields of pure and applied mathematics such as number theory, dynamical systems, probability, quantum mechanics, etc.

Recently, in [**2**], the authors considered the fractional version of the pantograph equation, namely

$$\text{(2)} \qquad \begin{cases} D^{\alpha}_{0+} u(t) = g\big(t, u(t), u(\lambda t)\big),\ t \in J = [0, T], \\ u(0) = u_0, \end{cases}$$

where $\alpha, \lambda \in (0, 1)$. The Banach contraction principle was the main tool used in this study.

The following hybrid differential of first order

$$\text{(3)} \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\Big[\dfrac{x(t)}{f(t, x(t))}\Big] = g\big(t, x(t)\big),\ t \in J = [0, T), \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

was studied by Dhage and Lakshmikantham [**9**], under the assumptions $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$.
In [**21**], Zhao et al. discussed the fractional version of Eq.(3), i.e.,

$$\text{(4)} \qquad \begin{cases} D^{\alpha}_{0+}\Big[\dfrac{x(t)}{f\big(t, x(t)\big)}\Big] = g\big(t, x(t)\big),\ t \in J,\ 0 < \alpha < 1, \\ x(0) = 0, \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$, a fixed point theorem in Banach algebras was the main tool used in this work.

In this paper, we study the following hybrid fractional pantograph equation

$$\text{(5)} \qquad \begin{cases} D^{\alpha}_{0+}\Big[\dfrac{x(t)}{f\big(t, x(t), x(\mu t)\big)}\Big] = g\big(t, x(t), x(\sigma t)\big),\ 0 < t < 1, \\ x(0) = 0, \end{cases}$$

where $0 < \alpha, \mu, \sigma < 1$, $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. The main tool in our study is a fixed point theorem for the product of two operators satisfying a condition of Darbo with respect to a measure of noncompactness. Moreover, we give some applications and examples where our results may be applied and we also compare these results with others appearing in the literature.

2. RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE AND INTEGRAL

We recall some definitions and results about fractional calculus theory from [**13**].

Definition 2.1. *The fractional Riemann-Liouville derivative of order* $\alpha > 0$ *of a continuous function* $h : (0, \infty) \to \mathbb{R}$ *is defined by*

$$D_{0+}^{\alpha} h(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x \frac{h(s)}{(x-s)^{\alpha-n+1}} \,\mathrm{d}s$$

provided that the right side is pointwise defined on $(0, \infty)$, *where* $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of* α.

Definition 2.2. *The Riemann-Liouville fractional integral of order* $\alpha > 0$ *of a function* $h : (0, \infty) \to \mathbb{R}$ *is defined by*

$$I_{0+}^{\alpha} h(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{h(s)}{(x-s)^{1-\alpha}} \,\mathrm{d}s,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.3. *Let* $h \in L^1(0,1)$ *and* $0 < \alpha < 1$. *Then*

(a) $D_{0+}^{\alpha} I_{0+}^{\alpha} h(x) = h(x)$

(b) $I_{0+}^{\alpha} D_{0+}^{\alpha} h(x) = h(x) - \dfrac{I_{0+}^{1-\alpha} h(x)|_{x=0}}{\Gamma(\alpha)} x^{\alpha-1}$ *a.e. on* $(0,1)$.

Lemma 2.4. *Let* $0 < \alpha < 1$ *and suppose that* $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ *and* $y \in C[0,1]$. *Then, the unique solution of the fractional hybrid initial value problem with linear modification of the argument*

$$\begin{cases} D_{0+}^{\alpha} \Big[\dfrac{x(t)}{f(t, x(t), x(\mu x))} \Big] = y(t), \ 0 < t < 1, \\ x(0) = 0, \end{cases} \tag{6}$$

is given by

$$x(t) = \frac{f(t, x(t), x(\mu x))}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}} \,\mathrm{d}s, \ t \in [0,1].$$

Proof. Let $x(t)$ be a solution of (6). Applying the operator I_{0+}^{α} to both sides of (6), taking into account Lemma 2.3, we obtain,

$$I_{0+}^{\alpha} D_{0+}^{\alpha} \left[\frac{x(t)}{f(t, x(t), x(\mu x))} \right] = I_{0+}^{\alpha} y(t),$$

or

$$\frac{x(t)}{f(t, x(t), x(\mu x))} - \frac{I_{0+}^{1-\alpha} \dfrac{x(t)}{f(t, x(t), x(\mu x))} \Big|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} = I_{0+}^{\alpha} y(t).$$

Since $\left.\dfrac{x(t)}{f(t,x(t),x(\mu x))}\right|_{t=0} = \dfrac{x(0)}{f(0,x(0),x(0))} = \dfrac{0}{f(0,0,0)} = 0$ (because $f(0,0,0) \neq 0$), we have

$$x(t) = \frac{f\big(t,x(t),x(\mu x)\big)}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}}\,\mathrm{d}s.$$

Conversely, suppose that $x(t)$ has the expression

$$x(t) = \frac{f\big(t,x(t),x(\mu x)\big)}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}}\,\mathrm{d}s,$$

or

$$x(t) = f\big(t,x(t),x(\mu x)\big) I^{\alpha}_{0+} y(t). \tag{7}$$

Applying D^{α}_{0+} on both sides of (7) after dividing both sides by $f\big(t,x(t),x(\mu x)\big)$, by the aid of Lemma 2.3, we obtain,

$$D^{\alpha}_{0+}\left[\frac{x(t)}{f(t,x(t),x(\mu x))}\right] = D^{\alpha}_{0+} I^{\alpha}_{0+} y(t) = y(t),\ 0 < t < 1.$$

Moreover, putting $t = 0$ in (7), we have $x(0) = f\big(0,x(0),x(0)\big) \cdot 0 = 0$. This proves that $x(t)$ is a solution of (6) which completes the proof.

3. MEASURE OF NONCOMPACTNESS

Assume that E is a real Banach space with the norm $\|.\|$ and the zero element θ. By $B(x,r)$ we denote the closed ball in E centered at x with radius r. By B_r we denote the ball $B(\theta,r)$. If X is a nonempty subset X of E then $\overline{X}$ and $\operatorname{Conv} X$ denote the closure and the convex closure of X, respectively. By $\operatorname{diam} X$ we denote the diameter of a bounded set X and $\|.\|$ is the norm of X, i.e., $\|X\| = \sup\{\|x\| : x \in X\}$. Further, by $\mathfrak{M}_E$ we denote the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets.

In this paper, we accept the following definition of measure of noncompactness [**3**].

Definition 3.5. *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0,\infty)$ will be called a measure of noncompactness in E if it satisfies the following conditions*:

1° *The family* $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ *is nonempty and* $\ker\mu \subset \mathfrak{N}_E$.

2° $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

3° $\mu(\overline{X}) = \mu(X)$.

4° $\mu(\operatorname{Conv} X) = \mu(X)$.

5° $\mu(\mu X + (1-\mu)Y) \leq \mu\, \mu(X) + (1-\mu)\, \mu(Y)$ *for* $\mu \in [0,1]$.

6° *If* (X_n) *is a sequence of closed subsets of* $\mathfrak{M}_E$ *such that* $X_{n+1} \subset X_n$ *and* $\lim_{n\to\infty} \mu(X_n) = 0$ *then* $X_\infty = \cap_{n=1}^{\infty} X_n \neq \phi$.

The family $\ker \mu$ appearing in 1° is called the kernel of the measure of noncompactness μ. Notice that the set X_∞ appearing in 6° belongs to $\ker\mu$. Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for any $n = 1, 2, \ldots$, we infer that $\mu(X_\infty) = 0$ which means that $X_\infty \in \ker \mu$.

In the sequel, we assume that the space E has structure of Banach algebra. We denote by xy the products of two elements $x, y \in E$ and by XY the set defined by $XY = \{xy : x \in X,\ y \in Y\}$.

Now, we recall the following concept which will play an important role in our considerations (see [**4**]).

Definition 3.6. *Let E be a Banach algebra. We say that a measure of noncompactness μ defined on E satisfies condition (m) if the following is satisfied*

$$\mu(XY) \leq \|X\|\mu(Y) + \|Y\|\mu(X), \text{ for any } X,\ Y \in \mathfrak{M}_E.$$

Aghajani et al. [**1**] proved the following generalization of fixed point theorem due to Darbo.

Theorem 3.7 (Theorem 2.2 of [**1**]). *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator satisfying*

$$\mu(TX) \leq \varphi(\mu(X)), \tag{8}$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \in \mathbb{R}_+$, where φ^n denotes the n-iteration of φ.

Then T has at least one fixed point in Ω.

Also, the authors of [**1**] proved the following lemma which will be useful in our considerations.

Lemma 3.8 (Lemma 2.1 of [**1**]). *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then, the following conditions are equivalent*:

(i) $\lim_{n\to\infty} \varphi^n(t) = 0$, *for any* $t \geq 0$

(ii) $\varphi(t) < t$, *for any* $t > 0$.

For convenience, we denote by $\mathcal{A}$ the class of functions given by

$$\mathcal{A} = \{\varphi : \mathbb{R}_+ \to \mathbb{R}_+ : \varphi \text{ is nondecreasing and } \lim_{n\to\infty} \varphi^n(t) = 0 \text{ for any } t > 0\},$$

where φ^n denotes the n-iteration of φ.

Remark 3.9. It is easy to see that if $\varphi \in \mathcal{A}$ then $\varphi(t) < t$, for any $t > 0$. Indeed, in contrary case, we can find $t_0 > 0$ and $t_0 \leq \varphi(t_0)$. By using the nondecreasing character of φ, we have

$$0 < t_0 \leq \varphi(t_0) \leq \varphi^2(t_0) \leq \ldots \leq \varphi^n(t_0) \leq \ldots,$$

and, consequently, $0 < t_0 \leq \lim_{n\to\infty} \varphi^n(t_0)$ which contradicts the fact that $\varphi \in \mathcal{A}$. Moreover, this proves that if $\varphi \in \mathcal{A}$ then φ is continuous at $t_0 = 0$.

Remark 3.10. By using Remark 3.9, the contractive condition appearing in Theorem 3.7 can be rewritten as $\mu(TX) < \mu(X)$ for any $X \in \mathfrak{M}_E \setminus \ker \mu$ and, therefore, Theorem 3.7 is a immediate consequence of Sadovskiĭ theorem [**19**].

In this paper, we work in the space $C[0,1]$ consisting of all real functions defined and continuous on the interval $[0,1]$ with the usual supremum norm

$$\|x\| = \sup\{|x(t)| : t \in [0,1]\},$$

for $x \in C[0,1]$. Notice that the space $C[0,1]$ is a Banach algebra, where the multiplication is defined as the usual product of real functions.

Next, we present the measure of noncompactness in $C[0,1]$ which will be used in our study. Let us fix a set $X \in \mathfrak{M}_{C[0,1]}$ and $\varepsilon > 0$. For $x \in X$, we denote by $\omega(x,\varepsilon)$ the modulus of continuity of x, i.e.,

$$\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0,1],\ |t-s| \leq \varepsilon\}.$$

Further, put

$$\omega(X,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon\to 0} \omega(X,\varepsilon).$$

In [**3**], it is proved that $\omega_0(X)$ is a measure of noncompactness in $C[0,1]$.

Proposition 3.11. *The measure of noncompactness ω_0 on $C[0,1]$ satisfies condition (m).*

Proof. Fix $X, Y \in \mathfrak{M}_{C[0,1]}$, $\varepsilon > 0$ and $t, s \in [0,1]$ with $|t-s| \leq \varepsilon$. Then, for $x \in X$ and $y \in Y$, we have,

$$\begin{aligned} |x(t)y(t) - x(s)y(s)| &\leq |x(t)y(t) - x(t)y(s)| + |x(t)y(s) - x(s)y(s)| \\ &= |x(t)|\,|y(t) - y(s)| + |y(s)|\,|x(t) - x(s)| \\ &\leq \|x\|\,\omega(y,\varepsilon) + \|y\|\,\omega(x,\varepsilon). \end{aligned}$$

This gives us,

$$\omega(xy,\varepsilon) \leq \|x\|\,\omega(y,\varepsilon) + \|y\|\,\omega(x,\varepsilon),$$

and, consequently,

$$\omega(XY,\varepsilon) \leq \|X\|\,\omega(Y,\varepsilon) + \|Y\|\,\omega(X,\varepsilon).$$

Taking $\varepsilon \to 0$, we get,

$$\omega_0(XY) \leq \|X\|\,\omega_0(Y) + \|Y\|\,\omega_0(X).$$

4. MAIN RESULTS

Here, we study (5) under the following assumptions:

(a_1) $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

(a_2) The functions f and g satisfy

$$|f(t, x_1, x_2) - f(t, \widehat{x}_1, \widehat{x}_2)| \leq \varphi_1 \left(\max(|x_1 - \widehat{x}_1|, |x_2 - \widehat{x}_2|)\right)$$

and

$$|g(t, x_1, x_2) - g(t, \widehat{x}_1, \widehat{x}_2)| \leq \varphi_2 \left(\max(|x_1 - \widehat{x}_1|, |x_2 - \widehat{x}_2|)\right),$$

respectively, for any $t \in [0,1]$ and $x_1, \widehat{x}_1, x_2, \widehat{x}_2 \in \mathbb{R}$, where $\varphi_1, \varphi_2 \in \mathcal{A}$ and φ_1 is continuous.

Notice that assumption (a_1) gives us the existence of two nonnegative constants k_1 and k_2 such that $|f(t,0,0)| \leq k_1$ and $|g(t,0,0)| \leq k_2$ for any $t \in [0,1]$.

(a_3) There exists $r_0 > 0$ satisfying the inequalities

$$(\varphi_1(r) + k_1) \cdot (\varphi_2(r) + k_2) \leq r\Gamma(\alpha + 1)$$

and

$$\varphi_2(r) + k_2 \leq \Gamma(\alpha + 1).$$

For further purposes, by Lemma 2.4 we have that any solution of (5) must satisfy the integral equation

$$x(t) = \frac{f(t, x(t), x(\mu t))}{\Gamma(\alpha)} \int_0^t \frac{g\big(s, x(s), x(\sigma s)\big)}{(t-s)^{1-\alpha}} \, \mathrm{d}s, \ 0 \leq t \leq 1.$$

Therefore, the fixed points of the operator $\mathcal{T}$ defined on $C[0,1]$ by

$$(\mathcal{T}x)(t) = \frac{f(t, x(t), x(\mu t))}{\Gamma(\alpha)} \int_0^t \frac{g\big(s, x(s), x(\sigma s)\big)}{(t-s)^{1-\alpha}} \, \mathrm{d}s, \ \ 0 \leq t \leq 1, \tag{9}$$

are the solutions of Eq.(5).

Theorem 4.12. *Under assumptions* (a_1) $-$ (a_3), (5) *has at least one solution in* $C[0,1]$.

Proof. Consider the operators $\mathcal{F}$ and $\mathcal{G}$ defined on $C[0,1]$ by

$$(\mathcal{F}x)(t) = f\big(t, x(t), x(\mu t)\big)$$

and

$$(\mathcal{G}x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\big(s, x(s), x(\sigma s)\big)}{(t-s)^{1-\alpha}} \mathrm{d}s$$

for any $x \in C[0,1]$ and $t \in [0,1]$. Therefore, $\mathcal{T}x = (\mathcal{F}x)\cdot(\mathcal{G}x)$, for any $x \in C[0,1]$.

For better readability, we divide the proof in several steps.

Step 1. $\mathcal{T} : C[0,1] \to C[0,1]$.

In fact, we prove that $\mathcal{F}x, \mathcal{G}x \in C[0,1]$ for $x \in C[0,1]$ and, since the product of continuous functions is a continuous function, our claim will be proved.

First, we prove that if $x \in C[0,1]$ then $\mathcal{F}x \in C[0,1]$. By assumption (a_1) and, since that $x \in C[0,1]$ then $x \circ \lambda \in C[0,1]$, where $\lambda : [0,1] \to [0,1]$ is given by $\lambda(t) = \mu t$, we infer that $\mathcal{F}x \in C[0,1]$. Now, we prove that if $x \in C[0,1]$ then $\mathcal{G}x \in C[0,1]$. We fix $t_0 \in [0,1]$ and let (t_n) be a sequence in $[0,1]$ such that $t_n \to t_0$. We have to prove that $(\mathcal{G}x)(t_n) \to (\mathcal{G}x)(t_0)$. In fact, without loss of generality we can suppose that $t_n > t_0$. Then, we have

$$\begin{aligned}|(\mathcal{G}x)(t_n) - (\mathcal{G}x)(t_0)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_n} \frac{g(s,x(s),x(\sigma s))}{(t_n-s)^{1-\alpha}}\,\mathrm{d}s - \int_0^{t_0} \frac{g(s,x(s),x(\sigma s))}{(t_0-s)^{1-\alpha}}\,\mathrm{d}s\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_n} \frac{g(s,x(s),x(\sigma s))}{(t_n-s)^{1-\alpha}}\,\mathrm{d}s - \int_0^{t_n} \frac{g(s,x(s),x(\sigma s))}{(t_0-s)^{1-\alpha}}\,\mathrm{d}s\right| \\ &\quad + \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_n} \frac{g(s,x(s),x(\sigma s))}{(t_0-s)^{1-\alpha}}\,\mathrm{d}s - \int_0^{t_0} \frac{g(s,x(s),x(\sigma s))}{(t_0-s)^{1-\alpha}}\,\mathrm{d}s\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^{t_n} \left|(t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right|\; |g(s,x(s),x(\sigma s))|\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_n} \left|(t_0-s)^{\alpha-1}\right|\; |g(s,x(s),x(\sigma s))|\,\mathrm{d}s.\end{aligned}$$

Since $g \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, g is bounded on $[0,1]\times[-\|x\|,\|x\|]\times[-\|x\|,\|x\|]$. Put $M = \sup\{|g(s,y_1,y_2)| : s \in [0,1],\ y_1,y_2 \in [-\|x\|,\|x\|]\}$. From the last estimate, we get

$$\begin{aligned}|(\mathcal{G}x)(t_n) - (\mathcal{G}x)(t_0)| &\le \frac{M}{\Gamma(\alpha)}\int_0^{t_n} \left|(t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right|\mathrm{d}s \\ &\quad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t_n} \left|(t_0-s)^{\alpha-1}\right|\mathrm{d}s.\end{aligned}$$

Since $0 < \alpha < 1$ and $t_n > t_0$, we have,

$$\begin{aligned}&|(\mathcal{G}x)(t_n) - (\mathcal{G}x)(t_0)| \le \frac{M}{\Gamma(\alpha)}\Bigg[\int_0^{t_0} \left|(t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right|\mathrm{d}s \\ &\quad + \int_{t_0}^{t_n} \left|(t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right|\mathrm{d}s\Bigg] + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t_n} \frac{1}{(s-t_0)^{1-\alpha}}\,\mathrm{d}s \\ &= \frac{M}{\Gamma(\alpha)}\Bigg[\int_0^{t_0} \left[(t_0-s)^{\alpha-1} - (t_n-s)^{\alpha-1}\right]\mathrm{d}s \\ &\quad + \int_{t_0}^{t_n} \frac{ds}{(t_n-s)^{1-\alpha}} + \int_{t_0}^{t_n} \frac{ds}{(s-t_0)^{1-\alpha}}\Bigg] + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t_n} \frac{1}{(s-t_0)^{1-\alpha}}\,\mathrm{d}s\end{aligned}$$

$$\le \frac{M}{\Gamma(\alpha+1)}\left[(t_n-t_0)^\alpha+t_0^\alpha-t_n^\alpha+(t_n-t_0)^\alpha+(t_n-t_0)^\alpha\right]+\frac{M}{\Gamma(\alpha+1)}(t_n-t_0)^\alpha$$
$$\le \frac{4M}{\Gamma(\alpha+1)}(t_n-t_0)^\alpha+\frac{M}{\Gamma(\alpha+1)}(t_0^\alpha-t_n^\alpha)<\frac{4M}{\Gamma(\alpha+1)}(t_n-t_0)^\alpha,$$

where we have used the fact that $t_0^\alpha - t_n^\alpha < 0$ in the last inequality. Since $t_n \to t_0$, the last estimate gives us that $(\mathcal{G}x)(t_n) \to (\mathcal{G}x)(t_0)$. Therefore, $\mathcal{G}x \in C[0,1]$. This proves that if $x \in C[0,1]$ then $\mathcal{T}x \in C[0,1]$.

Step 2. An estimate of $\|\mathcal{T}x\|$ for $x \in C[0,1]$.

Fix $x \in C[0,1]$ and $t \in [0,1]$. Then, taking into account our assumptions, we get,

$$\begin{aligned}
|(\mathcal{T}x)(t)| &= |(\mathcal{F}x)(t)|\,|(\mathcal{G}x)(t)| \\
&= |f(t,x(t),x(\mu t))|\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g\big(s,x(s),x(\sigma s)\big)}{(t-s)^{1-\alpha}}\,\mathrm{d}s\right| \\
&\le \big[\,|f(t,x(t),x(\mu t))-f(t,0,0)|+|f(t,0,0)|\,\big] \\
&\quad\times\left[\frac{1}{\Gamma(\alpha)}\left|\int_0^t \frac{g\big(s,x(s),x(\sigma s)\big)-g(s,0,0)}{(t-s)^{1-\alpha}}\,\mathrm{d}s+\int_0^t \frac{g(s,0,0)}{(t-s)^{1-\alpha}}\mathrm{d}s\right|\,\right] \\
&\le \frac{1}{\Gamma(\alpha)}[\varphi_1(\max(|x(t)|,|x(\mu t)|))+k_1]\Big[\int_0^t \frac{|g\big(s,x(s),x(\sigma s)\big)-g(s,0,0)|}{(t-s)^{1-\alpha}}\,\mathrm{d}s \\
&\quad+\int_0^t \frac{|g(s,0,0)|}{(t-s)^{1-\alpha}}\,\mathrm{d}s\Big] \\
&\le \frac{1}{\Gamma(\alpha)}[\varphi_1\left(\max(\|x\|,\|x\|)\right)+k_1]\left[\int_0^t \frac{\varphi_2\left(\max(|x(s)|,|x(\sigma s)|)\right)}{(t-s)^{1-\alpha}}\,\mathrm{d}s\right. \\
&\quad\left.+k_2\int_0^t \frac{\mathrm{d}s}{(t-s)^{1-\alpha}}\right] \\
&\le \frac{1}{\Gamma(\alpha)}[\varphi_1\left(\max(\|x\|,\|x\|)\right)+k_1]\cdot[\varphi_2\left(\max(\|x\|,\|x\|)\right)+k_2]\int_0^t\frac{\mathrm{d}s}{(t-s)^{1-\alpha}} \\
&\le \frac{1}{\Gamma(\alpha+1)}(\varphi_1(\|x\|)+k_1)\cdot(\varphi_2(\|x\|)+k_2).
\end{aligned}$$

Consequently,

$$\|\mathcal{T}x\| \le \frac{1}{\Gamma(\alpha+1)}[\varphi_1(\|x\|)+k_1]\cdot[\varphi_2(\|x\|)+k_2].$$

From assumption (a_3), it follows that the operator $\mathcal{T}$ transforms B_{r_0} into itself. Moreover, from the last estimates, we get

$$\|\mathcal{F}B_{r_0}\| \le \varphi_1(r_0)+k_1 \tag{10}$$

and

$$\|\mathcal{G}B_{r_0}\| \leq \frac{\varphi_2(r_0)+k_2}{\Gamma(\alpha+1)}. \tag{11}$$

Step 3. The operators $\mathcal{F}$ and $\mathcal{G}$ are continuous on the ball B_{r_0}.

First, we prove that $\mathcal{F}$ is continuous on B_{r_0}. To do this, we fix $\varepsilon > 0$ and we take $x, y \in B_{r_0}$ with $\|x-y\| \leq \varepsilon$. Then, for $t \in [0,1]$, we have,

$$\begin{aligned}
|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| &= |f(t,x(t),x(\mu t)) - f(t,y(t),x(\mu t))| \\
&\leq \varphi_1\left(\max(|x(t)-y(t)|, |x(\mu t), y(\mu t)|)\right) \\
&\leq \varphi_1\left(\max\left(\|x-y\|, \|x-y\|\right)\right) \leq \varphi_1\left(\|x-y\|\right) \leq \varphi_1(\varepsilon) < \varepsilon.
\end{aligned}$$

The last estimate proves that $\mathcal{F}$ is continuous on B_{r_0}, where we have used Remark 3.9.

Next, we prove that $\mathcal{G}$ is continuous on B_{r_0}. In order to do this, we fix $\varepsilon > 0$ and take $x, y \in B_{r_0}$ with $\|x-y\| \leq \varepsilon$. Then, for $t \in [0,1]$, we have,

$$\begin{aligned}
|(\mathcal{G}x)(t) - (\mathcal{G}y)(t)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^t \frac{g(s,x(s),x(\sigma s))}{(t-s)^{1-\alpha}}\,\mathrm{d}s - \int_0^t \frac{g(s,y(s),y(\sigma s))}{(t-s)^{1-\alpha}}\,\mathrm{d}s\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|g(s,x(s),x(\sigma s)) - g(s,y(s),y(\sigma s))|}{(t-s)^{1-\alpha}}\,\mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\varphi_2\left(\max(|x(s)-y(s)|, |x(\sigma s)-y(\sigma s)|)\right)}{(t-s)^{1-\alpha}}\,\mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)}\varphi_2\left(\max(\|x-y\|, \|x-y\|)\right)\int_0^t \frac{\mathrm{d}s}{(t-s)^{1-\alpha}} \\
&\leq \frac{1}{\Gamma(\alpha+1)}\varphi_2(\|x-y\|)t^{\alpha} \leq \frac{1}{\Gamma(\alpha+1)}\varphi_2(\varepsilon) < \frac{\varepsilon}{\Gamma(\alpha+1)},
\end{aligned}$$

where we have used the fact that $\varphi_2(\varepsilon) < \varepsilon$ (Remark 3.9). The last chain of inequalities gives us

$$\|\mathcal{G}x - \mathcal{G}y\| < \frac{\varepsilon}{\Gamma(\alpha+1)}$$

and this proves the continuity of the operator $\mathcal{G}$ on B_{r_0}. Finally, since $\mathcal{T} = \mathcal{F}\cdot\mathcal{G}$, it follows that $\mathcal{T}$ is continuous on B_{r_0}.

Step 4. For $\phi \neq X \subset B_{r_0}$, estimates of $\omega_0(\mathcal{F}X)$ and $\omega_0(\mathcal{G}X)$.

Fix $\varepsilon > 0$ and take $x \in X$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \leq \varepsilon$. Then,

$$\begin{aligned}
|(\mathcal{F}x)(t_1) - (\mathcal{F}x)(t_2)| &= |f(t_1,x(t_1),x(\mu t_1)) - f(t_2,x(t_2),x(\mu t_2))| \\
&\leq |f(t_1,x(t_1),x(\mu t_1)) - f(t_1,x(t_2),x(\mu t_2))|
\end{aligned}$$

$$+|f(t_1,x(t_2),x(\mu t_2))-f(t_2,x(t_2),x(\mu t_2))|$$
$$\leq \varphi_1\left(\max\left(|x(t_1)-x(t_2)|,|x(\mu t_1)-x(\mu t_2)|\right)\right)+\omega(f,\varepsilon)$$
$$\leq \varphi_1\left(\max\left(\omega(x,\varepsilon),\omega(x,\mu\varepsilon)\right)\right)+\omega(f,\varepsilon),$$

where $\omega(f,\varepsilon)$ denotes the quantity

$$\omega(f,\varepsilon)=\sup\{|f(t_1,x,y)-f(t_2,x,y)|:t_1,t_2\in[0,1],\ |t_1-t_2|\leq\varepsilon,\ x,y\in[-r_0,r_0]\}.$$

Therefore,

$$\omega(\mathcal{F}X,\varepsilon)\leq\varphi_1\left(\max(\omega(X,\varepsilon),\omega(X,\mu\varepsilon))\right)+\omega(f,\varepsilon),$$

and, since $f(t,x,y)$ is uniformly continuous on bounded subsets of $[0,1]\times\mathbb{R}\times\mathbb{R}$, $\omega(f,\varepsilon)\to 0$ as $\varepsilon\to 0$. From the last inequality and using the fact that φ_1 is continuous, we infer

$$\omega_0(\mathcal{F}X)\leq\varphi_1\big(\max(\lim_{\varepsilon\to 0}\omega(X,\varepsilon),\lim_{\varepsilon\to 0}\omega(X,\mu\varepsilon))\big)$$
$$=\varphi_1\big(\max(\omega_0(X),\omega_0(X))\big)=\varphi_1(\omega_0(X)). \tag{12}$$

Now, we estimate $\omega_0(\mathcal{G}X)$. Fix $\varepsilon>0$ and we take $x\in X$, $t_1,t_2\in[0,1]$ with $|t_1-t_2|\leq\varepsilon$. Without loss of generality, we can suppose that $t_1<t_2$, then

$$|(\mathcal{G}x)(t_1)-(\mathcal{G}x)(t_2)|=\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}\frac{g(s,x(s),x(\sigma s))}{(t_2-s)^{1-\alpha}}\,\mathrm{d}s-\int_0^{t_1}\frac{g(s,x(s),x(\sigma s))}{(t_1-s)^{1-\alpha}}\,\mathrm{d}s\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right||g(s,x(s),x(\sigma s))|\,\mathrm{d}s\right.$$
$$\left.+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|g(s,x(s),x(\sigma s))|\mathrm{d}s\right]$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]|g(s,x(s),x(\sigma s))|\,\mathrm{d}s\right.$$
$$\left.+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|g(s,x(s),x(\sigma s))|\,\mathrm{d}s\right]$$

Since $g(t,x)$ is continuous on $[0,1]\times\mathbb{R}\times\mathbb{R}$, it is bounded on the compact set $[0,1]\times[-r_0,r_0]\times[-r_0,r_0]$. Put $L=\sup\{|g(t,x,y)|:t\in[0,1],\ x,y\in[-r_0,r_0]\}$. Then, from the last inequality, it follows that,

$$|(\mathcal{G}x)(t_1)-(\mathcal{G}x)(t_2)|$$
$$\leq\frac{L}{\Gamma(\alpha)}\left[\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right]$$
$$\leq\frac{L}{\Gamma(\alpha+1)}\left[(t_2-t_1)^{\alpha}+t_1^{\alpha}-t_2^{\alpha}+(t_2-t_1)^{\alpha}\right]$$
$$\leq\frac{2L}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}\leq\frac{2L}{\Gamma(\alpha+1)}\varepsilon^{\alpha},$$

where we have used the fact that $t_1^\alpha - t_2^\alpha \le 0$. Therefore,

$$\omega(\mathcal{G}x, \varepsilon) \le \frac{2L}{\Gamma(\alpha+1)}\varepsilon^\alpha,$$

and this gives us

$$\omega(\mathcal{G}X, \varepsilon) \le \frac{2L}{\Gamma(\alpha+1)}\varepsilon^\alpha. \tag{13}$$

Taking $\varepsilon \to 0$, we get $\omega_0(\mathcal{G}X) = 0$.

Step 5. For $\phi \ne X \subset B_{r_0}$, an estimate of $\omega_0(\mathcal{T}X)$.

Taking into account Proposition 3.11, (9), (12) and (13), we have

$$\begin{aligned}\omega_0(\mathcal{T}X) &= \omega_0(\mathcal{F}X \cdot \mathcal{G}X) \le \|\mathcal{F}X\|\omega_0(\mathcal{G}X) + \|\mathcal{G}X\|\omega_0(\mathcal{F}X)\\ &\le \|\mathcal{F}B_{r_0}\|\omega_0(\mathcal{G}X) + \|\mathcal{G}B_{r_0}\|\omega_0(\mathcal{F}X)\\ &\le \frac{1}{\Gamma(\alpha+1)}(\varphi_2(r_0) + k_2)\cdot\varphi_1(\omega_0(X)).\end{aligned}$$

By assumption (a_3), $\varphi_2(r_0) + k_2 \le \Gamma(\alpha+1)$, and, therefore,

$$\frac{1}{\Gamma(\alpha+1)}(\varphi_2(r_0) + k_2)\cdot\varphi_1 \in \mathcal{A}$$

(it is easy to prove that if $\beta \in [0,1]$ and $\varphi \in \mathcal{A}$ then $\beta\varphi \in \mathcal{A}$).

Finally, by Theorem 3.7, the operator $\mathcal{T}$ has at least one fixed point in B_{r_0}. This completes the proof.

5. APPLICATIONS AND EXAMPLES

The nonoscillatory character of the problems the solutions of (5) is an interesting question in real world. It means that the solutions of (5) have a constant sign. In connection with this question, we notice that if $f(t,x,y)$ and $g(t,x,y)$ have the same constant sign (this means that $f(t,x,y) > 0$ and $g(t,x,y) \ge 0$ or $f(t,x,y) < 0$ and $g(t,x,y) \le 0$ for any $t \in [0,1]$ and $x, y \in \mathbb{R}$) and under assumptions of Theorem 4.12 then the solution $x(t)$ of (5) is nonnegative. This is due to the fact that the solutions of (5) satisfy the integral equation

$$x(t) = \frac{f(t, x(t), x(\mu t))}{\Gamma(\alpha)}\int_0^t \frac{g(s, x(s), x(\sigma s))}{(t-s)^{1-\alpha}}\,\mathrm{d}s,\ 0 \le t \le 1. \tag{14}$$

Moreover, the following proposition connects with the above mentioned question.

Proposition 5.13. *Under assumptions of Theorem* 4.12 *and suppose that* $g(t,x,y)$ *has constant sign and* $g(t,x,y) \ne 0$ *for* $t \in [0,1]$ *and* $x, y \in \mathbb{R}$ *then the solution* $x(t)$ *of* (5) *obtained by Theorem* 4.12 *verifies that* $x(t) \ne 0$ *for* $0 < t < 1$.

Proof. Assume the contrary, we can find $t^* \in (0,1)$ with $x(t^*) = 0$. Next, since $x(t)$ satisfies Eq.(14), we have,

$$0 = x(t^*) = \frac{f(t^*, 0, x(\mu t^*))}{\Gamma(\alpha)} \int_0^{t^*} \frac{g(s, x(s), x(\sigma s))}{(t^* - s)^{1-\alpha}}\, \mathrm{d}s.$$

Taking into account that $f(t,x,y) \neq 0$ for any $t \in [0,1]$ and $x, y \in \mathbb{R}$, we infer that

$$\int_0^{t^*} (t^* - s)^{\alpha-1}\, g(s, x(s), x(\sigma s))\, \mathrm{d}s = 0. \tag{15}$$

Since $g(t,x,y)$ has constant sign and $(t^* - s)^{\alpha-1} > 0$ for $s \in [0, t^*)$, we deduce that

$$g(s, x(s), x(\sigma s)) = 0 \text{ a.e. } s \in [0, t^*].$$

This contradicts the fact that $g(t,x,y) \neq 0$ for $(t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$ which completes the proof. □

The following corollary is an application of Bolzano's theorem.

Corollary 5.14. *Under assumptions of Proposition* 5.13, *the solution* $x(t)$ *of* (5) *obtained by Theorem* 4.12 *satisfies that* $x(t) > 0$ *for* $t \in (0,1)$ *or* $x(t) < 0$ *for* $t \in (0,1)$.

On the other hand, if we perturb the data function in (5)

$$\begin{cases} D_{0+}^{\alpha}\left[\dfrac{x(t)}{f(t, x(t), x(\mu t))}\right] = g(t, x(t), x(\sigma t)) + \eta(t),\ 0 < t < 1, \\ x(0) = 0, \end{cases} \tag{16}$$

where $0 < \alpha, \mu, \sigma < 1$, $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\eta \in C[0,1]$, the assumptions (a_1) and (a_2) of Theorem 4.12 are satisfied if $f(t,x,y)$ and $g(t,x,y)$ also satisfy them and only, we would have to check assumption (a_3). This fact makes that our theorem (Theorem 4.12) is very applicable.

Next, we present an example illustrating Theorem 4.12.

Example 5.15. Consider the following fractional hybrid problem

$$\begin{cases} D_{0+}^{\frac{1}{2}}\left[\dfrac{x(t)}{\frac{t}{4} + \ln\left(1 + \left|x\left(\frac{t}{3}\right)\right|\right) + \frac{1}{4}}\right] = \dfrac{1}{4} + \dfrac{1}{10}x(t) + \dfrac{1}{10}x\Big(\dfrac{t}{2}\Big),\ 0 < t < 1, \\ x(0) = 0. \end{cases} \tag{17}$$

Notice that this problem is a particular case of (5), where $\alpha = \dfrac{1}{2}$, $\mu = \dfrac{1}{3}$, $\sigma = \dfrac{1}{2}$, $f(t,x,y) = \dfrac{t}{4} + \ln(1 + |y|) + \dfrac{1}{4}$ and $g(t,x,y) = \dfrac{1}{4} + \dfrac{x}{10} + \dfrac{y}{10}$.

It is clear that the functions f and g satisfy (a_1) of Theorem 4.12 with $|f(t,0,0)| = \left|\dfrac{t}{4} + \dfrac{1}{4}\right|$ and $|g(t,0,0)| = \dfrac{1}{4}$. Therefore, $k_1 = \dfrac{1}{2}$ and $k_1 = \dfrac{1}{4}$. Moreover, for any x_1, x_2, $\widehat{x}_1$, $\widehat{x}_2 \in \mathbb{R}$ and any $t \in [0,1]$, we have

$$|f(t, x_1, x_2) - f(t, \widehat{x}_1, \widehat{x}_2)| = |\ln(1 + |x_2|) - \ln(1 + |\widehat{x}_2|)|$$

$$= \ln\left(\frac{1+|x_2|}{1+|\widehat{x}_2|}\right) = \ln\left(1+\frac{|x_2|-|\widehat{x}_2|}{1+|\widehat{x}_2|}\right)$$
$$\leq \ln\left(1+(|x_2|-|\widehat{x}_2|)\right) \leq \ln\left(1+|x_2-\widehat{x}_2|\right),$$

where, without loss of generality, we have taken $|x_2| > |\widehat{x}_2|$. In this case, $\varphi_1(t) = \ln(1+t)$ for $t \in \mathbb{R}_+$ and it is easy to see that $\varphi_1 \in \mathcal{A}$.

On the other hand, for any $x_1, x_2, \widehat{x}_1, \widehat{x}_2 \in \mathbb{R}$ and any $t \in [0,1]$, we have

$$\begin{aligned}|g(t,x_1,x_2) - g(t,\widehat{x}_1,\widehat{x}_2)| &= \left|\frac{1}{10}(x_1-\widehat{x}_1) + \frac{1}{10}(x_2-\widehat{x}_2)\right| \\ &\leq \frac{1}{10}\left(|x_1-\widehat{x}_1| + |x_2-\widehat{x}_2|\right) \\ &\leq \frac{1}{5}\left(\max\left(|x_1-\widehat{x}_1| + |x_2-\widehat{x}_2|\right)\right)\end{aligned}$$

and, therefore, we can take $\varphi_2(t) = \frac{1}{5}t$. It is clear that $\varphi_2 \in \mathcal{A}$. Moreover, φ_1 is obviously continuous.

Finally, the inequality appearing in assumption (a_3) of Theorem 4.12 has the form

$$\left(\ln(1+r)+\frac{1}{2}\right)\left(\frac{1}{5}r+\frac{1}{4}\right) \leq \Gamma\left(\frac{3}{2}\right)r.$$

The last inequality is satisfied for $r_0 = 1$, since

$$\left(\ln 2+\frac{1}{2}\right)\left(\frac{1}{5}+\frac{1}{4}\right) \cong 0.5373 \leq 0.88623 \cong \Gamma\left(\frac{3}{2}\right).$$

Moreover,

$$\frac{1}{5}+\frac{1}{4} = 0.45 < 0.88623 \cong \Gamma\left(\frac{3}{2}\right).$$

Now, by using Theorem 4.12, (17) has at least one solution $x \in C[0,1]$ such that $\|x\| \leq 1$.

6. COMPARISON WITH OTHER RESULTS

The authors in [**21**] studied the fractional hybrid differential equation

$$\begin{cases} D^{\alpha}_{0^+}\left[\frac{x(t)}{f\big(t,x(t)\big)}\right] = g\big(t,x(t)\big), \ a.e.\ t \in [0,1], \\ x(0) = 0, \end{cases} \tag{18}$$

under the following conditions:

(i) $f \in C([0,1]\times\mathbb{R}, \mathbb{R}\setminus\{0\})$ and $g \in C([0,1]\times\mathbb{R},\mathbb{R})$.

(ii) The function $x \to \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in [0,1]$.

(iii) There exists a constant $L > 0$ such that,

$$|f(t,x) - f(t,y)| \leq L|x-y|,$$

for every $t \in [0,1]$ and $x, y \in \mathbb{R}$.

(iv) There exists a function $h \in L^1([0,1], \mathbb{R}_+)$ such that,

$$|g(t,x)| \leq h(t) \text{ a.e } t \in [0,1], \text{ for all } x \in \mathbb{R}.$$

(v) $\dfrac{L\|h\|_{L^1}}{\Gamma(\alpha+1)} < 1.$

They proved the following result.

Theorem 6.16 (Theorem 3.1 [**21**]). *Under assumptions* (i) – (v), (18) *has a solution in* $C[0,1]$.

It is easy to see that (18) is a particular case of (5) when the functions $f(t,x,y)$ and $g(t,x,y)$ are independent of y.

Next, we present an example which cannot be treated by using Theorem 6.16 but it can be studied by Theorem 4.12.

Example 6.17. Consider the following fractional hybrid initial value problem

$$\begin{cases} D_{0^+}^{\frac{1}{2}}\left[\dfrac{x(t)}{\frac{1}{5}+\ln(1+|x(t)|)}\right] = \dfrac{1}{8} + \dfrac{1}{10}|x(t)|, \; 0<t<1, \\ x(0)=0. \end{cases} \tag{19}$$

(19) is a particular case of (5), where $\alpha = \dfrac{1}{2}$, $\mu=\sigma=0$, $f(t,x,y) = \dfrac{1}{5} + \ln(1+|x|)$ and $g(t,x,y) = \dfrac{1}{8} + \dfrac{1}{10}|x|$.

Obviously, the functions f and g satisfy assumption (a_1) of Theorem 4.12 and, moreover, $k_1 = \sup\{|f(t,0,0)|\} = \dfrac{1}{5}$ and $k_2 = \sup\{|g(t,0,0)|\} = \dfrac{1}{8}$.

By using a similar argument as in Example 5.15, we obtain

$$|f(t,x_1,x_2) - f(t,\widehat{x}_1,\widehat{x}_2)| = |\ln(1+|x_1|) - \ln(1+|\widehat{x}_1|)| \leq \ln(1+|x_1-\widehat{x}_1|)$$

and

$$|g(t,x_1,x_2) - g(t,\widehat{x}_1,\widehat{x}_2)| = \frac{1}{10}\,||x_1| - |\widehat{x}_1|| \leq \frac{1}{10}|x_1-\widehat{x}_1|$$

for any $x_1, x_2, \widehat{x}_1, \widehat{x}_2 \in \mathbb{R}$ and $t \in [0,1]$. Therefore, the functions f and g satisfy assumption (a_2) of Theorem 4.12 with $\varphi_1(t) = \ln(1+t)$ for $t \in \mathbb{R}_+$ and $\varphi_2(t) = \dfrac{t}{10}$ for $t \in \mathbb{R}_+$. It is easy to see that $\varphi_1, \varphi_2 \in \mathcal{A}$ and φ_1 is continuous.

For assumption (a_3) of Theorem 4.12, we have the inequality

$$\left(\ln(1+r) + \frac{1}{5}\right)\left(\frac{1}{10}r + \frac{1}{8}\right) \leq \Gamma\left(\frac{3}{2}\right) r$$

and this inequality is satisfied by $r_0 = 1$, since

$$\left(\ln 2 + \frac{1}{5}\right)\left(\frac{1}{10} + \frac{1}{8}\right) \cong 0.200958 \leq 0.88623 \cong \Gamma\left(\frac{3}{2}\right).$$

Moreover,

$$\frac{1}{10} + \frac{1}{8} \cong 0.225 < \Gamma\left(\frac{3}{2}\right) \cong 0.88623.$$

Therefore, by Theorem 4.12, (19) has a solution $x \in C[0,1]$ with $\|x\| \leq 1$.

Notice that problem (19) cannot be treated by Theorem 6.16 since the function $g(t,x,y) = \frac{1}{8} + \frac{1}{10}|x|$ does not satisfy assumption (iv).

In [**2**], the authors studied the nonlinear pantograph equation, Eq.(2), under the following assumptions:

(i) $\alpha, \lambda \in (0,1)$.

(ii) $f : J \times X \times X \to X$ is a continuous function, where $(X, \|.\|)$ is a Banach space.

(iii) There exists a positive constant $L > 0$ such that

$$\|g(t,u,x) - g(t,v,y)\| \leq L(\|u-v\| + \|x-y\|),$$

for any $t \in J$ and $u, v, x, y \in X$.

(iv) $4\gamma L < 1$, where $\gamma = \dfrac{T^\alpha}{\Gamma(\alpha+1)}$.

They proved the following result.

Theorem 6.18 (Theorem 3.1 of [**2**]). *Under assumptions* (i)$-$(iv), (2) *has a unique solution in* $C(J \times X)$.

Next, we present an example which cannot be treated by Theorem 6.18 while it can be studied by Theorem 4.12.

Example 6.19. Consider the following fractional pantograph equation

$$\begin{cases} D_{0^+}^{\frac{1}{2}} x(t) = \dfrac{1}{25} + \ln\left(1 + \left|x\left(\dfrac{t}{2}\right)\right|\right), \ 0 < t < 1, \\ x(0) = 0. \end{cases} \tag{20}$$

(20) is a particular case of (5), where $\alpha = \frac{1}{2}$, $f(t,x,y) = 1$, $\sigma = \frac{1}{2}$ and $g(t,x,y) = \frac{1}{25} + \ln(1+|y|)$.

It is clear that the functions f and g satisfy assumption (a_1) of Theorem 4.12 and, moreover, $k_1 = \sup\{|f(t,0,0)| : t \in [0,1]\} = 1$ and $k_2 = \sup\{|g(t,0,0)| : t \in [0,1]\} = \frac{1}{25}$.

It is obvious that $\varphi_1(t) = 0$ and if $|y| > |y_1|$ then, by using a similar argument that in Example 5.15, we have

$$|g(t,x,y) - g(t,x_1,y_1)| = |\ln(1+|y|) - \ln(1+|y_1|)| \leq \ln(1+|y-y_1|).$$

Therefore, $\varphi_2(t) = \ln(1+t)$ and $\varphi_2 \in \mathcal{A}$.
In this case, assumption (a_3) of Theorem 4.12 is given by

$$\ln(1+r) + \frac{1}{25} \leq r\Gamma\left(\frac{3}{2}\right)$$

and it is easily seen that this inequality is satisfied by $r_0 = 1$, since

$$\ln 2 + \frac{1}{25} \cong 0.73344 < 0.88623 \cong \Gamma\left(\frac{3}{2}\right).$$

Therefore, by Theorem 4.12, problem (20) has a solution $x \in [0,1]$ with $\|x\| \le 1$.

On the other hand, an application of the mean value theorem gives us

$$|g(t,x,y) - g(t,x_1,y_1)| = |\ln(1+|y|) - \ln(1+|y_1|)| \le ||y| - |y_1|| \le |y - y_1|.$$

Consequently, $L = 1$ in assumption (iii) of Theorem 6.18.

Moreover, $\gamma = \dfrac{1}{\Gamma\left(\frac{3}{2}\right)}$ and $4\gamma L = \dfrac{4}{\Gamma\left(\frac{3}{2}\right)} = 4.51 > 1$.

Notice that the constant $L = 1$ cannot be improved in the last inequality since $(\ln(1+x))' = \dfrac{1}{1+x}$ for $x \ge 0$ and $\dfrac{1}{1+x}$ tends to 1 when x tends to zero.

Therefore, since $4\gamma L = 4.51 > 1$, problem (20) cannot be studied by Theorem 6.18.

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