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NORM INEQUALITIES FOR NEW CONVOLUTIONS AND THEIR APPLICATIONS

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Various L_p-weighted norm inequalities for some new types of convolutions are proved which generalize some known results on convolution norm inequalities. Applications are made in the field of integral transforms and differential equations.

1. INTRODUCTION

Integral inequalities are a basic tool in the study of qualitative as well as quantitative properties of integral transforms and solutions of differential equations. In particular, convolution inequalities are essential and in fact indispensable because a wide variety of integral transforms and solutions of differential equations are represented as convolutions (see [**4**], [**17**], [**18**]). Among all convolution type transformations certainly the best known one is Fourier convolution. For integrable functions on $\mathbb{R}$ the Fourier convolution of two functions is given by

$$(f *_1 g)(\eta) = \int_{\mathbb{R}} f(\xi) g(\eta - \xi) \mathrm{d}\xi, \quad \eta \in \mathbb{R}.$$

Carefully applying Hölder's inequality, one can obtain Young's inequality for Fourier convolution:

$$\|f *_1 g\|_{L_r(\mathbb{R})} \leq \|f\|_{L_p(\mathbb{R})} \|g\|_{L_q(\mathbb{R})}, \tag{1}$$

where $1 \leq p, q, r \leq \infty, 1/r = 1/p + 1/q - 1,\ f \in L_p(\mathbb{R})$ and $g \in L_q(\mathbb{R})$.

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In [**20**], by considering the L_p−norms in weighted spaces, S. Saitoh gave convolution norm inequality in the following form:

$$\left\|(f\rho_1 *_1 g\rho_2)(\rho_1 *_1 \rho_2)^{1/p-1}\right\|_{L_p(\mathbb{R})} \leq \left\|f\rho_1^{1/p}\right\|_{L_p(\mathbb{R})}\left\|g\rho_2^{1/p}\right\|_{L_p(\mathbb{R})}. \tag{2}$$

For practical application purposes inequality (2) has been improved as well as generalized in several different directions [**5–7, 10–13, 15, 16, 21–23**].

Recently, L. P. Castro and S. Saitoh [**3**] considered the Fourier convolution and additionally introduced the following three types of convolutions:

$$(F_1 *_2 F_2)(\eta) = \int_{\mathbb{R}} F_1(\xi)\overline{F_2(\xi-\eta)}\mathrm{d}\xi,$$
$$(F_1 *_3 F_2)(\eta) = \int_{\mathbb{R}} \overline{F_1(\xi)}F_2(\eta+\xi)\mathrm{d}\xi,$$
$$(F_1 *_4 F_2)(\eta) = \int_{\mathbb{R}} \overline{F_1(\xi)F_2(-\xi-\eta)}\mathrm{d}\xi,$$

for which they obtained new convolution inequalities that have some important and fundamental applications to the related integral equations containing corresponding types of convolutions as integral kernels. The method they used in the proofs is based on the theory of reproducing kernels [**2**] (see also [**19**]). However, they did not give the equality conditions for these inequalities.

Quite recently, in view of Hölder's inequality the authors [**14**] obtained some norm inequalities for Mellin convolutions which have been successfully implemented in many L_p−weighted estimates for integral transforms and solutions of partial differential equations.

Motivated by these works, in this paper we would like to combine the methods suggested by authors in [**14**] and [**15**] with the ideas of Castro and Saitoh in [**3**] to introduce some new convolution norm inequalities which generalize some results given in [**3**] and [**14**]. We also demonstrate the usefulness of our results to derive the boundedness of Hardy and Meijer transforms and obtain L_p−weighted estimates for solutions of certain differential equations.

Throughout the paper, let D be a finite or infinite interval of the real axis $\mathbb{R}$ and let $p, q > 1$ be conjugate exponents: $1/p+1/q = 1$. For $F_j(.) : D \to \mathbb{C},\ j = 1, 2,$ and $\varphi(.,.) : D \times D \to D$, we consider the following four types of convolutions:

$$(F_1 *_{1,\varphi} F_2)(\eta) = \int_D F_1(\xi)F_2(\varphi(\xi,\eta))\left|\frac{\partial\varphi}{\partial\eta}(\xi,\eta)\right|\mathrm{d}\xi, \tag{3}$$

$$(F_1 *_{2,\varphi} F_2)(\eta) = \int_D F_1(\xi)\overline{F_2(\varphi(\xi,\eta))}\left|\frac{\partial\varphi}{\partial\eta}(\xi,\eta)\right|\mathrm{d}\xi, \tag{4}$$

$$(F_1 *_{3,\varphi} F_2)(\eta) = \int_D \overline{F_1(\xi)}F_2(\varphi(\xi,\eta))\left|\frac{\partial\varphi}{\partial\eta}(\xi,\eta)\right|\mathrm{d}\xi, \tag{5}$$

$$(F_1 *_{4,\varphi} F_2)(\eta) = \int_D \overline{F_1(\xi)F_2(\varphi(\xi,\eta))}\left|\frac{\partial\varphi}{\partial\eta}(\xi,\eta)\right|\mathrm{d}\xi, \tag{6}$$

where $\left|\frac{\partial\varphi}{\partial\eta}(\xi,\eta)\right|$ is the Jacobian of the transformation $\eta \mapsto \varphi(.,\eta)$. For a (non-negative and measurable) weight ρ on D, let us denote by $L_p(D,\rho)$ the Lebesgue space of complex-valued measurable functions F on D such that $\|F\|_\rho < \infty$ on the support of ρ, where

$$\|F\|_\rho := \left(\int_D \frac{|F(x)|^p}{\rho(x)}\,\mathrm{d}x\right)^{1/p},$$

and $F = 0$ on the outside of the support of ρ.

2. NORM INEQUALITIES FOR NEW CONVOLUTIONS

Our main theorem is the following.

Theorem 1. *Let $\varphi_k : D\times D \to D$, $k = 1,2,3,4$. Let $\rho_j, j = 1,2$, be suitable weights such that there exists a new weight ρ:*

$$\rho(\eta) = \left[\sum_{k=1}^{4}\left(\rho_1^{q/p} *_{k,\varphi_k} \rho_2^{q/p}\right)(\eta)\right]^{p/q}, \quad \eta \in D.$$

Then, for $F_j \in L_p(D,\rho_j), j = 1,2$, we have $\sum_{k=1}^{4}(F_1 *_{k,\varphi_k} F_2) \in L_p(D,\rho)$, and moreover,

$$\left\|\sum_{k=1}^{4}(F_1 *_{k,\varphi_k} F_2)\right\|_\rho^p \leq 4\|F_1\|_{\rho_1}^p\|F_2\|_{\rho_2}^p. \tag{7}$$

Proof. First, for $F_j \in L_p(D,\rho_j), j = 1,2$, and $k = 1,2,3,4$, we have

$$\begin{aligned}
&\left|(F_1 *_{k,\varphi_k} F_2)(\eta)\right| \\
&\leq \int_D |F_1(\xi)|\left|F_2(\varphi_k(\xi,\eta))\right|\left|\frac{\partial\varphi_k}{\partial\eta}(\xi,\eta)\right|\mathrm{d}\xi \\
&= \int_D \frac{|F_1(\xi)|}{\rho_1^{1/p}(\xi)}\frac{|F_2(\varphi_k(\xi,\eta))|}{\rho_2^{1/p}(\varphi_k(\xi,\eta))}\left|\frac{\partial\varphi_k}{\partial\eta}(\xi,\eta)\right|^{1/p}\left|\frac{\partial\varphi_k}{\partial\eta}(\xi,\eta)\right|^{1/q}\rho_1^{1/p}(\xi)\rho_2^{1/p}(\varphi_k(\xi,\eta))\mathrm{d}\xi,
\end{aligned}$$

that, by Hölder's inequality, yields

$$\begin{aligned}
&\left|(F_1 *_{k,\varphi_k} F_2)(\eta)\right| \\
&\leq \left[\left(\rho_1^{q/p} *_{1,\varphi_k} \rho_2^{q/p}\right)(\eta)\right]^{1/q}\left[\left(\left(\frac{|F_1|^p}{\rho_1}\right) *_{1,\varphi_k}\left(\frac{|F_2|^p}{\rho_2}\right)\right)(\eta)\right]^{1/p}
\end{aligned} \tag{8}$$

for all $\eta \in D$. Therefore,

$$\begin{aligned}
&\Big|\sum_{k=1}^{4}(F_1 *_{k,\varphi_k} F_2)(\eta)\Big|^p\\
&\leq \left\{\sum_{k=1}^{4}\Big[\big(\rho_1^{q/p} *_{1,\varphi_k} \rho_2^{q/p}\big)(\eta)\Big]^{1/q}\Big[\Big(\Big(\frac{|F_1|^p}{\rho_1}\Big) *_{1,\varphi_k} \Big(\frac{|F_2|^p}{\rho_2}\Big)\Big)(\eta)\Big]^{1/p}\right\}^p\\
&\leq \Big[\sum_{k=1}^{4}\big(\rho_1^{q/p} *_{1,\varphi_k} \rho_2^{q/p}\big)(\eta)\Big]^{p/q}\sum_{k=1}^{4}\Big(\Big(\frac{|F_1|^p}{\rho_1}\Big) *_{1,\varphi_k} \Big(\frac{|F_2|^p}{\rho_2}\Big)\Big)(\eta),
\end{aligned}$$

which holds in view of Hölder's inequality for all $\eta \in D$. This implies that

$$\frac{\Big|\sum_{k=1}^{4}(F_1 *_{k,\varphi_k} F_2)(\eta)\Big|^p}{\rho(\eta)} \leq \sum_{k=1}^{4}\Big(\Big(\frac{|F_1|^p}{\rho_1}\Big) *_{1,\varphi_k} \Big(\frac{|F_2|^p}{\rho_2}\Big)\Big)(\eta), \quad \eta \in D. \tag{9}$$

Integrating both sides of (9) with respect to η over D, using Fubini's theorem and substituting the variables in integrals, we obtain inequality (7) as required. □

In particular, for each type of convolutions we obtain the following corollary.

Corollary 2. *Let $\varphi_k : D \times D \to D$, $k = 1,2,3,4$. Let $\rho_j, j = 1,2$, be suitable weights such that there exists a new weight ρ:*

$$\rho(\eta) = \Big[\big(\rho_1^{q/p} *_{k,\varphi_k} \rho_2^{q/p}\big)(\eta)\Big]^{p/q}, \quad \eta \in D.$$

Then, for $F_j \in L_p(D,\rho_j), j = 1,2,$ we have

$$\|F_1 *_{k,\varphi_k} F_2\|_\rho \leq \|F_1\|_{\rho_1}\|F_2\|_{\rho_2}. \tag{10}$$

In Corollary 2, let us consider $D = \mathbb{R}$ and

$$\begin{aligned}
\varphi_1(\xi,\eta) &= \eta - \xi,\\
\varphi_2(\xi,\eta) &= \xi - \eta,\\
\varphi_3(\xi,\eta) &= \xi + \eta,\\
\varphi_4(\xi,\eta) &= -\xi - \eta,
\end{aligned}$$

for $(\xi,\eta) \in \mathbb{R}\times\mathbb{R}$. Then, we obtain the following corollaries.

Corollary 3. *For $F_j \in L_p(\mathbb{R},\rho_j), j = 1,2,$ we have*

$$\|F_1 *_k F_2\|_\rho \leq \|F_1\|_{\rho_1}\|F_2\|_{\rho_2}, \tag{11}$$

where

$$\rho(\eta) = \Big[\big(\rho_1^{q/p} *_k \rho_2^{q/p}\big)(\eta)\Big]^{p/q}, \quad \eta \in \mathbb{R},$$

for $k = 1,2,3,4$.

We notice here that inequality (11) is sharp; that is, there exist extremal functions for which equality between norms is attained:

$$k=1,\quad F_1(\xi)=C_1e^{\alpha\xi+i\beta\xi}\rho_1^{q/p}(\xi),\quad F_2(\xi)=C_2e^{\alpha\xi+i\beta\xi}\rho_2^{q/p}(\xi)\quad \text{a.e. on } \mathbb{R},$$
$$k=2,\quad F_1(\xi)=C_1e^{\alpha\xi+i\beta\xi}\rho_1^{q/p}(\xi),\quad F_2(\xi)=C_2e^{-\alpha\xi+i\beta\xi}\rho_2^{q/p}(\xi)\quad \text{a.e. on } \mathbb{R},$$
$$k=3,\quad F_1(\xi)=C_1e^{-\alpha\xi+i\beta\xi}\rho_1^{q/p}(\xi),\quad F_2(\xi)=C_2e^{\alpha\xi+i\beta\xi}\rho_2^{q/p}(\xi)\quad \text{a.e. on } \mathbb{R},$$
$$k=4,\quad F_1(\xi)=C_1e^{-\alpha\xi+i\beta\xi}\rho_1^{q/p}(\xi),\quad F_2(\xi)=C_2e^{-\alpha\xi+i\beta\xi}\rho_2^{q/p}(\xi)\quad \text{a.e. on } \mathbb{R},$$

for complex constants C_j and real numbers α,β such that $F_j\in L_p(\mathbb{R},\rho_j), j=1,2$. Indeed, for $k=1$, we refer the reader to [**1**] for details. For $k=2$, equality in (11) implies that equality holds in (8) for a.e. $\eta\in\mathbb{R}$. Unless $F_j=0$ a.e. on $\mathbb{R}$, this happens only if for a.e. $\eta\in\mathbb{R}$, there exists a complex number $G(\eta)$ such that

$$F_1(\xi)\rho_1^{-q/p}(\xi)\overline{F_2(\xi-\eta)\rho_2^{-q/p}(\xi-\eta)}=G(\eta)\quad \text{a.e.}\quad \xi\in\mathbb{R}. \tag{12}$$

In the same way as in [**1**], we can prove that G is measurable on $\mathbb{R}$. Put

$$f_1(\xi)=F_1(\xi)\rho_1^{-q/p}(\xi),\quad f_2(\xi)=\overline{F_2(-\xi)\rho_2^{-q/p}(-\xi)},\quad \xi\in\mathbb{R}.$$

We see that $f_j, j=1,2$, are measurable on $\mathbb{R}$ and for a.e $\eta\in\mathbb{R}$,

$$f_1(\xi)f_2(\eta-\xi)=G(\eta)\quad \text{a.e}\quad \xi\in\mathbb{R}. \tag{13}$$

It follows from (13) that

$$f_1(\xi)f_2(\eta)=G(\xi+\eta)\quad \text{a.e}\quad \xi\in\mathbb{R},\quad \eta\in\mathbb{R}, \tag{14}$$

which, by [**1**, Lemma], yields

$$f_j(\xi)=C_je^{\alpha\xi+i\beta\xi}\quad \text{a.e.}\quad \xi\in\mathbb{R},$$

for complex constants $C_j\neq 0, j=1,2$, and fixed $\alpha,\beta\in\mathbb{R}$. Hence, we deduce that

$$F_1(\xi)=C_1e^{\alpha\xi+i\beta\xi}\rho_1^{q/p}(\xi),\quad F_2(\xi)=C_2e^{-\alpha\xi+i\beta\xi}\rho_2^{q/p}(\xi)\quad \text{a.e. on } \mathbb{R}$$

as required. Similar arguments apply to the cases $k=3$ and $k=4$.

Corollary 4. *For $F_j\in L_p(\mathbb{R},\rho_j), j=1,2$, we have the following inequality*

$$\left\|\sum_{k=1}^{4}(F_1 *_k F_2)\right\|_\rho^p \le 4\|F_1\|_{\rho_1}^p\|F_2\|_{\rho_2}^p, \tag{15}$$

provided there exists

$$\rho(\eta):=\left[\sum_{k=1}^{4}\left(\rho_1^{q/p} *_k \rho_2^{q/p}\right)(\eta)\right]^{p/q},\quad \eta\in\mathbb{R}.$$

Equality holds in (15) *if, and only if,*

$$F_j(\xi) = C_j e^{i\beta\xi} \rho_j^{q/p}(\xi) \quad \text{a.e. on } \mathbb{R} \tag{16}$$

for complex constants C_j *and fixed* $\beta \in \mathbb{R}$ *such that* $F_j \in L_p(\mathbb{R}, \rho_j), j = 1, 2$.

Corollaries 3 and 4 generalize some corresponding results of Castro and Saitoh [**3**], which are useful in the study of the integral equation with the mixed Toeplitz-Hankel kernel

$$\lambda\varphi(x) + \int_{\mathbb{R}} \big[k_1(x-y) - k_2(x+y)\big]\varphi(y)\mathrm{d}y = f(x). \tag{17}$$

For wider applicability of the results, let us now consider some more special cases of the convolution given in (3). In (3), by putting $D = (0,\infty) := \mathbb{R}_+$ and

$$\varphi_1(\xi,\eta) = \xi\eta, \quad (\xi,\eta) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

we obtain the Mellin convolution type:

$$(F_1 \diamond F_2)(\eta) := \int_{\mathbb{R}_+} F_1(\xi) F_2(\xi\eta)\xi \mathrm{d}\xi. \tag{18}$$

Then, in view of Corollary 2, we have the following result.

Corollary 5. *For two weights* ρ_1 *and* ρ_2 *such that there exists a new weight*

$$\rho(\eta) = \Big[\big(\rho_1^{q/p} \diamond \rho_2^{q/p}\big)(\eta)\Big]^{p/q}, \quad \eta \in \mathbb{R}_+,$$

and for $F_j \in L_p(\mathbb{R}_+, \rho_j), j = 1, 2$, *we have the following inequality*

$$\|F_1 \diamond F_2\|_\rho \le \|F_1\|_{\rho_1} \|F_2\|_{\rho_2}. \tag{19}$$

The equality holds here if, and only if,

$$F_1(\xi) = C_1 \xi^{-\alpha} \rho_1^{q/p}(\xi), \quad F_2(\xi) = C_2 \xi^{\alpha} \rho_2^{q/p}(\xi) \quad \text{a.e. on } \mathbb{R}_+ \tag{20}$$

for complex constants C_j *and real number* α *such that* $F_j \in L_p(\mathbb{R}_+, \rho_j), j = 1, 2$.

In many cases, the following Mellin convolution type is considered:

$$(F_1 \circ F_2)(\eta) := \int_{\mathbb{R}_+} F_1(\xi) F_2(\xi\eta) \mathrm{d}\xi. \tag{21}$$

By an argument analogous to that used for the proof of Theorem 1, we obtain the following corollary.

Corollary 6. *For $F_j \in L_p(\mathbb{R}_+, \rho_j), j = 1, 2,$ we have the following inequality*

$$\|F_1 \circ F_2\|_\tau \le \|F_1\|_{\rho_1} \|F_2\|_{\rho_2}, \tag{22}$$

provided there exists

$$\tau(\eta) = \Big[\big(\rho_3^{q/p} \circ \rho_2^{q/p}\big)(\eta)\Big]^{p/q}, \quad \eta \in \mathbb{R}_+,$$

where $\rho_3(\xi) = \xi^{-1}\rho_1(\xi)$. The equality holds here if, and only if,

$$F_1(\xi) = C_1 \xi^{-q/p-\alpha} \rho_1^{q/p}(\xi), \quad F_2(\xi) = C_2 \xi^{\alpha} \rho_2^{q/p}(\xi) \quad \text{a.e. on } \mathbb{R}_+ \tag{23}$$

for complex constants C_j and real number α such that $F_j \in L_p(\mathbb{R}_+, \rho_j), j = 1, 2.$

Finally, for fixed function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\psi'(\xi) > 0$ for all $\xi \in \mathbb{R}_+$, let us consider the convolution

$$(F_1 \star F_2)(\eta) := \int_0^\eta F_1(\xi) F_2(\psi(\eta) - \psi(\xi)) \psi'(\eta) \mathrm{d}\xi.$$

We observe that the Laplace convolution is a special case of the $\star$ convolution.

Corollary 7. *Let ρ_1 and ρ_2 be two weights such that there exists a new weight*

$$\rho(\eta) = \Big[\big(\rho_1^{q/p} \star \rho_2^{q/p}\big)(\eta)\Big]^{p/q}, \quad \eta \in \mathbb{R}_+.$$

Then, for $F_j \in L_p(\mathbb{R}_+, \rho_j), j = 1, 2,$ we have $F_1 \star F_2 \in L_p(\mathbb{R}_+, \rho)$, and moreover,

$$\|F_1 \star F_2\| \le \|F_1\|_{\rho_1} \|F_2\|_{\rho_2}. \tag{24}$$

Equality holds in (24) if, and only if,

$$F_1(\xi) = C_1 e^{\alpha\psi(\xi) + i\beta\psi(\xi)} \rho_1^{q/p}(\xi), \quad F_2(\xi) = C_2 e^{\alpha\xi + i\beta\xi} \rho_2^{q/p}(\xi) \quad \text{a.e. on } \mathbb{R}_+ \tag{25}$$

for complex constants C_j and real numbers α, β such that $F_j \in L_p(\mathbb{R}_+, \rho_j), j = 1, 2.$

The proofs of equality statements in Corollaries 5, 6 and 7 are similar to that of Corollary 3 so the details are omitted.

Of course, the convolutions in Corollaries 5, 6 and 7 can be replaced by others that are particular cases of the convolutions given in (4), (5) and (6).

3. APPLICATIONS

In this section, we show some typical applications. We first study the boundedness of Hardy and Meijer transforms. Then, we give L_p-weighted estimates for solutions of certain differential equations.

EXAMPLE 1. We consider the Hardy transform (see [**8**])

$$\mathcal{H}_\nu\{F\}(x) = \int_0^\infty tC_\nu(xt)F(t)\mathrm{d}t, \quad \nu > -1, \tag{26}$$

where

$$C_\nu(t) = \cos(\alpha\pi)J_\nu(t) + \sin(\alpha\pi)Y_\nu(t), \quad \alpha \in \mathbb{R},$$

$J_\nu(t)$ and $Y_\nu(t)$ are the Bessel functions of the first and second kinds, respectively.

The asymptotic representations of J_ν and Y_ν near zero and infinity are given by ([**9**, pp. 33-34])

$$J_\nu(x) = \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^\nu [1 + O(x)] \quad (x \to 0),$$
$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left[\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right] \quad (x \to \infty)$$

and

$$Y_\nu(x) = -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu [1 + O(x)] \quad (x \to 0),$$
$$Y_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left[\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right] \quad (x \to \infty),$$

respectively. Consequently, for $p\nu < 1 < p/2$, we see that $C_\nu \in L_p(\mathbb{R}_+)$.

Then, for $p\nu < 1 < p/2$, and a weight ρ such that

$$\int_{\mathbb{R}_+} x\rho^{q/p}(x)\mathrm{d}x < \infty, \tag{27}$$

Corollary 5 implies that the Hardy transform $\mathcal{H}_\nu$ is a bounded operator from $L_p(\mathbb{R}_+, \rho)$ to $L_p(\mathbb{R}_+)$, and moreover,

$$\int_{\mathbb{R}_+} |\mathcal{H}_\nu\{F\}(x)|^p \mathrm{d}x \leq \left[\int_{\mathbb{R}_+} x\rho^{q/p}(x)\mathrm{d}x\right]^{p/q} \int_{\mathbb{R}_+} |C_\nu(x)|^p \mathrm{d}x \int_{\mathbb{R}_+} \frac{|F(x)|^p}{\rho(x)}\mathrm{d}x, \tag{28}$$

provided $F \in L_p(\mathbb{R}_+, \rho)$.

Notice that for $\alpha = 0$ the Hardy transform coincides with one of the forms of the Hankel transform, and for $\alpha = 1/2$ with the $Y-$transform. So, inequality (28) also gives the boundedness of these transforms.

EXAMPLE 2. We now consider the Meijer transform (see [**18**, Chapter 7])

$$\mathcal{M}_\nu\{F\}(x) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}_+} \sqrt{xt}K_\nu(xt)F(t)\mathrm{d}t, \tag{29}$$

where $K_\nu(x)$ denotes the McDonald function which is defined via the modified Bessel functions by

$$K_\nu(x) = \frac{\pi}{2\sin(\pi\nu)}\left[\cos(\pi\nu)I_{-\nu}(x) - I_\nu(x)\right], \quad \nu \in \mathbb{R}\setminus\mathbb{Z}.$$

Using the asymptotic behavior of $K_\nu(x)$ near zero and infinity ([**9**, p. 36]),

$$K_\nu(x) = \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu [1 + O(x)] \quad (x \to 0, \nu > 0)$$

and

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left[1 + O\Big(\frac{1}{x}\Big)\right] \quad (x \to \infty)$$

we observe that $\sqrt{x}K_\nu(x)$ belongs to $L_p(\mathbb{R}_+)$ provided $2+p-2p\nu > 0$. Hence, for a weight ρ such that

$$\int_{\mathbb{R}_+}\left(\frac{\rho(x)}{x}\right)^{q/p}\mathrm{d}x < \infty,$$

$F \in L_p(\mathbb{R}_+, \rho)$ and $2 + p - 2p\nu > 0$, in view of inequality (22), we get

$$\begin{aligned} &\int_{\mathbb{R}_+}|\mathcal{M}_\nu\{F\}(x)|^p\mathrm{d}x \\ &\le \left(\frac{2}{\pi}\right)^{p/2}\left[\int_{\mathbb{R}_+}\left(\frac{\rho(x)}{x}\right)^{q/p}\mathrm{d}x\right]^{p/q}\int_{\mathbb{R}_+}|\sqrt{x}K_\nu(x)|^p\mathrm{d}x\int_{\mathbb{R}_+}\frac{|F(x)|^p}{\rho(x)}\mathrm{d}x, \end{aligned} \tag{30}$$

which implies that the Meijer transform $\mathcal{M}_\nu$ is a bounded operator from $L_p(\mathbb{R}_+, \rho)$ to $L_p(\mathbb{R}_+)$.

Furthermore, the Meijer transform $\mathcal{M}_\nu$ is also a bounded operator from $L_2(\mathbb{R}_+)$ to $L_2(\mathbb{R}_+)$ if $2 - 2\nu > \alpha > 0$. Indeed, let

$$\rho(x) = x^\alpha e^{-x}, \quad \alpha > 0, \quad x \in \mathbb{R}_+.$$

Then, we see that $\rho(x) \in L_1(\mathbb{R}_+, x)$ and $\sqrt{x}K_\nu(x) \in L_2(\mathbb{R}_+, \rho)$. Hence, applying (22) to the Meijer transform $\mathcal{M}_\nu\{F\}$ yields

$$\int_{\mathbb{R}_+}|\mathcal{M}_\nu\{F\}(x)|^2\mathrm{d}x \le \frac{2}{\pi}\int_{\mathbb{R}_+}\frac{\rho(x)}{x}\mathrm{d}x\int_{\mathbb{R}_+}\frac{|\sqrt{x}K_\nu(x)|^2}{\rho(x)}\mathrm{d}x\int_{\mathbb{R}_+}|F(x)|^2\mathrm{d}x \tag{31}$$

provided $F \in L_2(\mathbb{R}_+)$.

Example 3. We consider the axisymmetric boundary value problem

$$\nabla^4 u(x,t) = 0, \quad 0 \le x < \infty, \quad t > 0, \tag{32}$$

with the boundary data

$$u(x,0) = f(x), \quad 0 \le x < \infty, \tag{33}$$

$$\frac{\partial u}{\partial t} = 0 \quad \text{on } t = 0, \quad 0 \le x < \infty, \tag{34}$$

$$u(x,t) \to 0 \quad \text{as } x \to \infty, \tag{35}$$

where the axisymmetric biharmonic operator is

$$\nabla^4 = \nabla^2(\nabla^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial t^2}\right).$$

The use of the Hankel transform to this problem gives the formal solution (see [**4**, p. 328])

$$u(x,t)=\int_{\mathbb{R}_+}(1+t\xi)e^{-t\xi}F(\xi)J_0(x\xi)\xi\mathrm{d}\xi, \tag{36}$$

where F is the Hankel transform of f:

$$F(\xi)=\int_{\mathbb{R}_+}J_0(\xi\eta)f(\eta)\eta\mathrm{d}\eta,\quad \xi\in\mathbb{R}_+.$$

Let ρ be a weight such that

$$\int_{\mathbb{R}_+}\rho^{q/p}(\xi)\xi\mathrm{d}\xi<\infty$$

and let $f\in L_p(\mathbb{R}_+,\rho), p>2$. Then, inequality (28) asserts that $F\in L_p(\mathbb{R}_+)$, and moreover,

$$\int_{\mathbb{R}_+}|F(\xi)|^p\mathrm{d}\xi\le\left[\int_{\mathbb{R}_+}\rho^{q/p}(\xi)\xi\mathrm{d}\xi\right]^{p/q}\int_{\mathbb{R}_+}|J_0(\xi)|^p\mathrm{d}\xi\int_{\mathbb{R}_+}\frac{|f(\xi)|^p}{\rho(\xi)}\mathrm{d}\xi. \tag{37}$$

Now, as an application of (19) we obtain

$$\int_{\mathbb{R}_+}|u(x,t)|^p\mathrm{d}x\le\left[\int_{\mathbb{R}_+}\frac{(1+t\xi)^q}{e^{qt\xi}}\xi\mathrm{d}\xi\right]^{p/q}\int_{\mathbb{R}_+}|J_0(\xi)|^p\mathrm{d}\xi\int_{\mathbb{R}_+}|F(\xi)|^p\mathrm{d}\xi. \tag{38}$$

From (37) and (38) we claim that the solution $u(x,t)$ satisfies the following L_p-estimate

$$\begin{aligned}\int_{\mathbb{R}_+}|u(x,t)|^p\mathrm{d}x\le&\left[\int_{\mathbb{R}_+}\frac{(1+t\xi)^q}{e^{qt\xi}}\xi\mathrm{d}\xi\int_{\mathbb{R}_+}\rho^{q/p}(\xi)\xi\mathrm{d}\xi\right]^{p/q}\\&\times\left[\int_{\mathbb{R}_+}|J_0(\xi)|^p\mathrm{d}\xi\right]^2\int_{\mathbb{R}_+}\frac{|f(\xi)|^p}{\rho(\xi)}\mathrm{d}\xi.\end{aligned} \tag{39}$$

Example 4. Let $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be such that $\psi'(x)>0$ for all $x\in\mathbb{R}_+$. Let us consider the following integral equation

$$y(x)+\psi'(x)\int_0^x y(t)\mathrm{d}t=F(x), \tag{40}$$

where F is given function. Some special cases of the above equation can be found in [**18**].

Put

$$z(x)=\int_0^x y(t)\mathrm{d}t,\quad x\in\mathbb{R}_+.$$

Then, solving the original integral equation is reduced to solving the first-order linear nonhomogeneous ordinary differential equation

$$z'(x)+\psi'(x)z(x)=F(x) \tag{41}$$

under the initial condition

$$\lim_{x\to0}z(x)=0.$$

Therefore, the solution of (40) is given by

$$y(x)=F(x)-\psi'(x)\int_0^x F(t)\exp\left(\psi(t)-\psi(x)\right)\mathrm{d}t. \tag{42}$$

Let ρ be a weight such that there exists a new weight

$$\omega(x) = \left[\psi'(x) \int_0^x \rho^{q/p}(t)\mathrm{d}t\right]^{p/q}, \quad x \in \mathbb{R}_+,$$

and $F \in L_p(\mathbb{R}_+, \rho) \cap L_p(\mathbb{R}_+, \omega)$. Then, from (24) and Minkowski's inequality, we obtain the following estimate

$$\|y\|_\omega \leq \|F\|_\omega + p^{-1/p}\|F\|_\rho. \tag{43}$$

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