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COMPLETE SOLUTION TO SEVEN-POINT SCHEMES OF DISCRETE ANISOTROPIC LAPLACIAN ON REGULAR HEXAGONS

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We consider in this work hexagonal seven-point schemes for two-dimensional diagonal-form Laplacian. Ordinary and compact seven-point stencils are investigated by even-odd decompositions. The result gives explicit complete solutions to both ordinary and compact hexagonal seven-point schemes for all phase angles.

1. INTRODUCTION

Hexagonal finite volumes (Hex FVs) are of interest in some recent works. In the study of the origin of U-wave in ECG (electrocardiogram) [**1**], hexagonal subregions are adopted, but only pure algebraic relations are considered. In numerical simulation [**2**] of human heart electrophysiology, Hex FV based finite difference (FD) methods are developed and analyzed, leading to discrete ordinary and compact seven-point schemes of the standard Laplacian with applications to Poisson equations and reaction-diffusion systems. The work [**2**] showed advantages of Hex FVs over square grids in studies of wave phenomena propagated in curved domains, while the theoretic and numerical investigations were extended in a separate work to semi-linear Poisson equations involving anisotropic Laplacian on nets of regular hexagons of two specific orientations. In the current work, the schemes for anisotropic Laplacian are further generalized to regular hexagons of all orientations. The basic theory is as complete as possible. Things not discussed in this paper include, among others, numerical examples of PDEs, grid generations and

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practical issues about boundary values. In some limitted scope, these can be found in our previous work.

Aiming at two-dimensional applications involving anisotropic Laplacian (e.g., [**5**]), we consider discretization on hexagons for solving the linear partial differential equation involving *diagonal-form* Laplacian,

$$\text{Lap}\; u \equiv \text{L}u := D_1 u_{xx} + D_2 u_{yy} = f(x,y), \tag{1}$$

with positive constants D_1 and D_2. The applications include both static and time-dependent problems. For the current work, we confine our discussion to the equation subject to Dirichlet type boundary condition on a net of (regular) hexagons with an arbitrary phase angle. We introduce regular hexagons in general configurations together with the associated local geometry and neighborhood topology. Discretizations based on both ordinary and compact seven-point schemes are analyzed.

In two-dimensional applications of configurations consisting of (subset of) Cartesian type regular hexagons, we denote the radius of hexagons by r, the height by $h\Big(=\dfrac{\sqrt{3}}{2}r\Big)$, and the center-to-center distance by $d(=2h)$. Near a typical center node, $P_0 = (x_0, y_0)$, the six neighbor (center) nodes are

$$P_j = (x_j, y_j) = (x_0, y_0) + d(\cos\xi_j, \sin\xi_j), \;\; \xi_j = \varphi + \frac{j\pi}{3} + \frac{\pi}{6}, \;\; 1 \le j \le 6. \tag{2}$$

Here the *phase angle*, φ, is the configurarion parameter. Two particular instances are called type I ($\varphi = 0$) and type II ($\varphi = -\pi/6$) for convenience. Hexagon centers in lattices of these two types are indexed as for an orthogonal Cartesian mesh as shown in Table 1, while the geometry and neighborhood of a general Hex FV shown in Table 2. Indexing rules are illustrated in Figs. 1(a), 1(b), and 2(a), 2(b).

For convenience, we abuse the notations and denote FV centers in a neighborhood (Figs. 2(a), 2(b)) by an *ordered list*,

$$\begin{aligned} \text{Type I}: \;\; \{P_j\}_{j=0}^{6} &= \{P,\; P_{N},\; P_{NW},\; P_{SW},\; P_{S},\; P_{SE},\; P_{NE}\}, \\ \text{Type II}: \;\; \{P_j\}_{j=0}^{6} &= \{P,\; P_{NE},\; P_{NW},\; P_{W},\; P_{SW},\; P_{NW},\; P_{E}\}. \end{aligned} \tag{3}$$

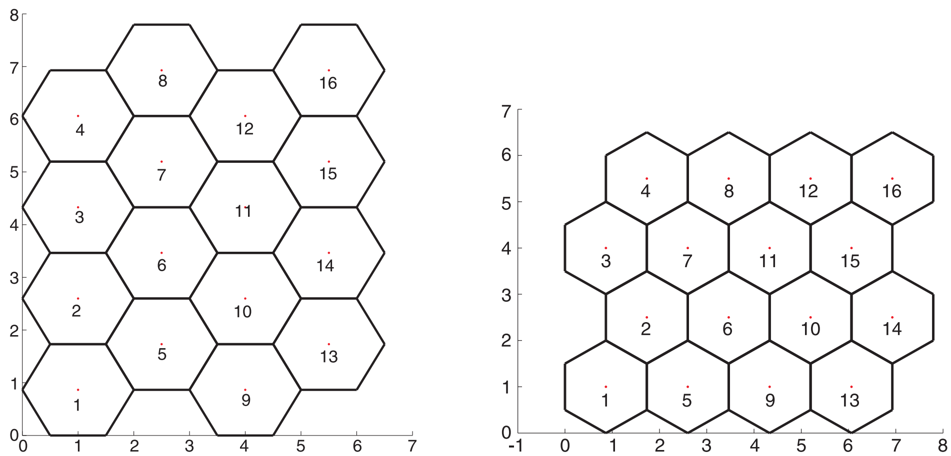
We note for applications that a two-dimensional irregular domain may be approximated by a sequence of (not necessarily Cartesian) nets of hexagons. Actually, our work in numerical modeling of ECG depends on this (Algorithm 1 in ([**2**])).

Phase angle	Type I, $\varphi = 0$		Type II, $\varphi = -\pi/6$	
Center point	i^{even}	i^{odd}	j^{even}	j^{odd}
$cx(i,j)$	$(1.5i - 0.5)r$		$2ih$	$(2i-1)h$
$cy(i,j)$	$2jh$	$(2j-1)h$	$(1.5j - 0.5)r$	

Table 1. Lattices of type I and II regular hexagons.

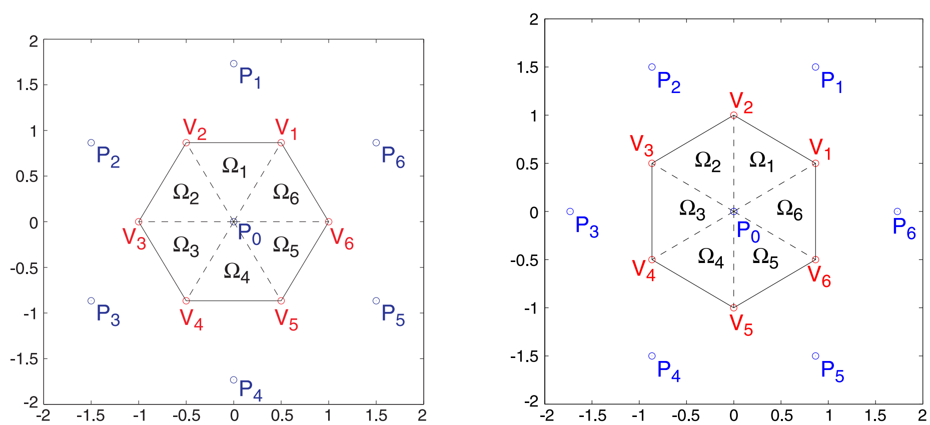
Phase angle	$\varphi \in \mathbb{R}$
Vertices	$V_k = (vx(*,k), vy(*,k)), \quad k = 1, 2, \cdots, 6.$
$vx(*,k)$	$cx(*) + r\cos(\varphi + \frac{k\pi}{3})$
$vy(*,k)$	$cy(*) + r\sin(\varphi + \frac{k\pi}{3})$
Neighbor centers	$P_k = V_k + V_{k+1} - P_0, \qquad k = 1, 2, \cdots, 6.$

Table 2. Local geometry at a regular hexagon : six vertices and neighbor centers.



(a) Type I lattice (b) Type II lattice

Figure 1. Lattice of regular hexagons in natural order by columns



(a) Type I neighborhood (b) Type II neighborhood

Figure 2. Regular hexagonal FV neighborhood.

In the special case of solving an isotropic Laplacian (Eq.(1) with $D_1 = D_2 = D$), the divergence theorem is applied to an integral version of the equation, in obviously interpreted notations,

$$\iint_{\Omega} f \,\mathrm{d}x\,\mathrm{d}y = \iint_{\Omega} \nabla\cdot(D_1u_x, D_2u_y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} \overrightarrow{n}\cdot(D_1u_x, D_2u_y)\,\mathrm{d}\gamma$$
$$= \sum_{i=1}^{6} \int_{\overline{V_iV_{i+1}}} \overrightarrow{n_i}\cdot(D_1u_x, D_2u_y)\,\mathrm{d}\gamma_i = D\sum_{i=1}^{6}\int_{\overline{V_iV_{i+1}}} \frac{\partial u}{\partial\overrightarrow{n_i}}\,\mathrm{d}\gamma_i \tag{4}$$

Discretization of each of the line integrals by mid-point rule implies the construction of an ordinary hexagonal seven-point scheme. The locally second-order quadrature actually yields a (global) second-order approximation, and a fourth-order compact scheme is obtained with more effort [**2**].

However, the analytic part of the argument does not apply directly in case of a general anisotropic Laplacian ($D_1 \neq D_2$), for which the intuition would still suggest the application of the mid-point or the trapezoid rule to each line integral. One needs then *proper* estimates of the gradient at FV vertices or mid-points of edges in terms of values of u at several local center nodes. Instead, we take an algebraic approach in the current work to extend the previous results for all phase angles φ and positive D_1, D_2.

We note that, (i) spectral analysis of iterative methods solving the discrete anisotropic Laplacian on a net of hexagons seems not as easy as the analysis on square grids, since finite trigonometric series is incomplete for the error analysis (even) on a single regular hexagon ([**3**],[**4**]), and (ii) for Hex FVs of types I and II, convergence of schemes developed in the current work were analyzed via numerical investigation together with matrix properties by kind of comparison in a separate work.

As for the remaining sections, basics and main results of hexagonal stencils are introduced in Section 2, while ordinary seven-point scheme detailed in Section 3, and compact seven-point scheme in Section 4. Briefly discussed in Section 5 is a cell-vertex FV approach. Conclusion is given in the final section. The appendices include some barycentric relations in Appendix A and a *formal* proof of convergence with leading error terms of the ordinary seven-point method in Appendix B.

2. DISCRETE HEXAGONAL SEVEN-POINT LAPLACIAN

At a Hex FV, a compact hexagonal seven-point (H7c) scheme refers to the discrete algebraic expression,

$$\sum_{j=0}^{6} A_j u_{P_j} = \sum_{j=0}^{6} R_j (\mathrm{L}u)_{P_j}. \tag{5}$$

An ordinary seven-point (H7) scheme corresponds to the special case that $R_0 = 1, R_1 = R_2 = \cdots = R_6 = 0$. The stencils $\{A_j, R_j\}_{j=0}^{6}$ are constants, hopefully determined by interpolation conditions that the above defining equations hold exact on a properly chosen subset of low-degree *shifted* monomials (for slightly better numerical stability), $\{(x-x_0)^m(y-y_0)^n \mid 0 \le m,n,\ m+n \le 6\}$, and by a normalization constraint,

$$\sum_{j=0}^{6} R_j = 1, \quad \text{or} \quad \text{equivalently}, \quad R_0 = 1 - \sum_{j=1}^{6} R_j, \tag{6}$$

which is needed for uniqueness because of scale-invariance of Eq.(5).

We note the (*half-turn*) *symmetry* in the local geometry,

$$P_{j+3} - P_0 = -(P_j - P_0), \quad j = 1,2,3. \tag{7}$$

Without loss of generalities, we assume $P_0 = (x_0, y_0) = (0,0)$ and consider standard monomials. By this reduction, the corresponding interpolation conditions are

$$\sum_{j=0}^{6} A_j x_j^m y_j^n = \sum_{j=0}^{6} R_j \big(D_1 m(m-1) x_j^{m-2} y_j^n + D_2 n(n-1) x_j^m y_j^{n-2} \big),\ m,n \ge 0, \tag{8}$$

up to proper interpretation. An immediate consequence,

$$A_0 = -\sum_{j=1}^{6} A_j\ , \tag{9}$$

is implied by the *first* interpolant ($m = n = 0, u(x,y) = 1$). All other cases ($m+n > 0$) actually involve only $\{A_1, A_2, ..., A_6\}$. In view of Eqs.(6,9), we focus mostly on $\{A_j, R_j\}_{j=1}^{6}$ in the rest of the discussion.

The seven-point methods (H7 and H7c) are expected to be at least of second-order. We collect in Table 3 the interpolation conditions on monomials of degrees up to four for easy reference. Although, our software is implemented in a unified manner using SVD or LU, based on Eq.(8).

Refering to Table 3, we observe the following.

1. For an H7 scheme, the stencil $\{R_j\}_1^6$ is trivial and the RHS (Lu-term) vanishes on almost all monomials except $(m,n) \in \{(2,0),(0,2)\}$.
2. For an H7c with symmetric $R_j's$ (Eq.(10)), the RHS vanishes on cubic monomials, and $\{R_j\}_{j=1}^{6}$ appear explicitly only if $m+n \ge 4$.

2.1. Even-odd decomposition of stencils

A general H7c scheme consists of 14 coefficients. In addition to the normalization constraint, we need 13 (independent) interpolation conditions.

Case	m,n	u	Lu	LHS	RHS
1	0,0	1	0	$\sum_0^6 A_j$	0
2	1,0	x	0	$\sum_1^3 (A_j - A_{j+3})x_j$	0
3	0,1	y	0	$\sum_1^3 (A_j - A_{j+3})y_j$	0
4	2,0	x^2	$2D_1$	$\sum_1^3 (A_j + A_{j+3})x_j^2$	$2D_1 \sum_0^6 R_j = 2D_1$
5	0,2	y^2	$2D_2$	$\sum_1^3 (A_j + A_{j+3})y_j^2$	$2D_2 \sum_0^6 R_j = 2D_2$
6	1,1	xy	0	$\sum_1^3 (A_j + A_{j+3})x_j y_j$	0
7	3,0	x^3	$6D_1 x$	$\sum_1^3 (A_j - A_{j+3})x_j^3$	$6D_1 \sum_1^3 (R_j - R_{j+3})x_j$
8	0,3	y^3	$6D_2 y$	$\sum_1^3 (A_j - A_{j+3})y_j^3$	$6D_2 \sum_1^3 (R_j - R_{j+3})y_j$
9	2,1	$x^2 y$	$2D_1 y$	$\sum_1^3 (A_j - A_{j+3})x_j^2 y_j$	$2D_1 \sum_1^3 (R_j - R_{j+3})y_j$
10	1,2	xy^2	$2D_2 x$	$\sum_1^3 (A_j - A_{j+3})x_j y_j^2$	$2D_2 \sum_1^3 (R_j - R_{j+3})x_j$
11	4,0	x^4	$12D_1 x^2$	$\sum_1^3 (A_j + A_{j+3})x_j^4$	$12D_1 \sum_1^3 (R_j + R_{j+3})x_j^2$
12	0,4	y^4	$12D_2 y^2$	$\sum_1^3 (A_j + A_{j+3})y_j^4$	$12D_2 \sum_1^3 (R_j + R_{j+3})y_j^2$
13	3,1	$x^3 y$	$6D_1 xy$	$\sum_1^3 (A_j + A_{j+3})x_j^3 y_j$	$6D_1 \sum_1^3 (R_j + R_{j+3})x_j y_j$
14	1,3	xy^3	$6D_2 xy$	$\sum_1^3 (A_j + A_{j+3})x_j y_j^3$	$6D_2 \sum_1^3 (R_j + R_{j+3})x_j y_j$
15	2,2	$x^2 y^2$	$2D_1 y^2 + 2D_2 x^2$	$\sum_1^3 (A_j + A_{j+3})x_j^2 y_j^2$	$2\sum_1^3 (R_j + R_{j+3})(D_2 x_j^2 + D_1 y_j^2)$

Table 3. Interpolation conditions (Eq.(8)) on monomials for hexagonal seven-point methods. Stated expressions depend only on that $x_0 = y_0 = 0$ and the geometric symmetry $(x_{j+3}, y_{j+3}) = -(x_j, y_j)$, $j = 1, 2, 3$. The algebraic system decouples accordingly, up to the even-odd decompositions of stencils. For an H7 scheme, $R_0 = 1, R_1 = R_2 = \cdots = R_6 = 0$.

In view of the geometric symmetry (Eq.(7)), we expect *symmetric* stencils, that is, calculations *would* produce a (unique) solution satisfying

$$A_{j+3} = A_j, \quad R_{j+3} = R_j, \qquad j = 1, 2, 3. \tag{10}$$

The prediction is nearly correct, as will be stated in the main result shortly.

Definition 1. *The (even) symmetric and the asymmetric parts of the stencils* $\{A_j\}_0^6$ *and* $\{R_j\}_0^6$ *are, respectively,*

$$A^{sym} := \Big\{A_0, \frac{A_j + A_{j+3}}{2}\Big\} \equiv \Big\{A_0, \frac{A_1 + A_4}{2}, \frac{A_2 + A_5}{2}, \frac{A_3 + A_6}{2}, \frac{A_4 + A_1}{2}, \frac{A_5 + A_2}{2}, \frac{A_6 + A_3}{2}\Big\},$$

$$R^{sym} := \Big\{R_0, \frac{R_j + R_{j+3}}{2}\Big\} \equiv \Big\{R_0, \frac{R_1 + R_4}{2}, \frac{R_2 + R_5}{2}, \frac{R_3 + R_6}{2}, \frac{R_4 + R_1}{2}, \frac{R_5 + R_2}{2}, \frac{R_6 + R_3}{2}\Big\},$$

$$A^{asym} := \Big\{0, \frac{A_j - A_{j+3}}{2}\Big\} \equiv \Big\{0, \frac{A_1 - A_4}{2}, \frac{A_2 - A_5}{2}, \frac{A_3 - A_6}{2}, \frac{A_4 - A_1}{2}, \frac{A_5 - A_2}{2}, \frac{A_6 - A_3}{2}\Big\},$$

$$R^{asym} := \Big\{*, \frac{R_j - R_{j+3}}{2}\Big\} \equiv \Big\{*, \frac{R_1 - R_4}{2}, \frac{R_2 - R_5}{2}, \frac{R_3 - R_6}{2}, \frac{R_4 - R_1}{2}, \frac{R_5 - R_2}{2}, \frac{R_6 - R_3}{2}\Big\},$$

with the value R_0^{asym} *not of our concern.*

For convenience, we also refer to the triple $\{A_j + A_{j+3}\}_{j=1,3,5}$ as the symmetric part and $\{A_j - A_{j+3}\}_{j=1,3,5}$ the asymmetric part. Similarly with the R_j's.

Easily justified properties are

1. $A = A^{sym} + A^{asym}, \quad R = R^{sym} + R^{asym}$.

2. A^{sym} is symmetric, i.e., $A_j^{sym} = A_{j+3}^{sym},\ j = 1, 2, 3$. Same with R^{sym}.

3. A^{asym} is asymmetric, i.e., $A_j^{asym} = -A_{j+3}^{asym},\ j = 1, 2, 3$. So is R^{asym}.

4. If $\{A_j\}_1^6$ is both symmetric and asymmetric, then $\{A_j\}_0^6 \equiv 0$. Therefore *the even-odd* decomposition of $\{A_j\}_0^6$ is unique.

5. If $\{R_j\}_1^6$ is both symmetric and asymmetric, then $\{R_j\}_1^6 \equiv 0$ and $R_0 = 1$. The H7c scheme reduces to an H7 scheme.

We state the main result. All cases of interpolation conditions refer to Table 3.

(a) The symmetric part A^{sym} is uniquely solvable by cases $\{x^2, y^2, xy\}$ in terms of only $\{D_1, D_2, d, \varphi\}$, as in Lemma 3 later.

(b) With A^{sym} available, R^{sym} is uniquely solvable by using monomials $\{x^4, y^4, x^3y\}$ or $\{x^4, y^4, xy^3\}$. Different solution may be obtained accordingly, as to be explained in Lemma 6 (Eq.(22)) and Remark 2.

(c) It is necessary to assume $R = R^{sym}(,\ R^{asym} = 0)$ for uniqueness of the coupled $\{A_j's,\ R_j's\}$, as will be detailed in Subsections 4.1 and 4.2, esp., Remark 1.

(d) $A^{asym}(= 0)$ is uniquely determined, as will be shown in Lemma 2 for H7 and in Lemma 5 for H7c, by using R^{asym} (zero or a nonzero constant).

We note the only source of non-uniqueness in the final solution, $A(= A^{sym} + A^{asym})$ and $R(= R^{sym} + R^{asym})$, is the R^{asym} term (if not *null*).

The *half-turn symmetry* (Eq.(7)) of the local geometry at a Hex FV readily implies the following, for both H7 and H7c.

Lemma 1. *Assuming (even) symmetry in both stencils (Eq. (10)), the interpolation conditions are satisfied on all odd monomials. That is, for* $u(x, y) = x^m y^n$,

$$\sum_{j=1}^{6} A_j x_j^m y_j^n = 0 = \sum_{j=1}^{6} R_j (Lu)_{P_j}, \qquad \text{if} \quad \text{m+n} \quad \text{is} \quad \text{odd}.$$

3. ORDINARY SEVEN-POINT SCHEME

We discuss the H7 scheme first. The linear system of interpolation conditions consisting of $u \in \{x^2, y^2, xy, x, y, x^3, y^3\}$ reads

$$(11)\qquad \begin{bmatrix} x_1^2 & x_3^2 & x_5^2 & 0 & 0 & 0 \\ y_1^2 & y_3^2 & y_5^2 & 0 & 0 & 0 \\ x_1y_1 & x_3y_3 & x_5y_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & x_3 & x_5 \\ 0 & 0 & 0 & y_1 & y_3 & y_5 \\ 0 & 0 & 0 & x_1^3 & x_3^3 & x_5^3 \\ 0 & 0 & 0 & y_1^3 & y_3^3 & y_5^3 \end{bmatrix} \cdot \begin{bmatrix} A_1 + A_4 \\ A_3 + A_6 \\ A_5 + A_2 \\ A_1 - A_4 \\ A_3 - A_6 \\ A_5 - A_2 \end{bmatrix} = \begin{bmatrix} 2D_1 \\ 2D_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We rewrite this decoupled system as

$$\begin{bmatrix} M_{sym} & 0 \\ 0 & M_{asym} \end{bmatrix} \cdot \begin{bmatrix} A^{sym} \\ A^{asym} \end{bmatrix} = (2D_1, 2D_2, 0, 0, 0, 0, 0)^T,$$

with respectively the symmetric and asymmetric parts,

$$(12)\qquad M_{sym} \cdot A^{sym} \equiv \begin{bmatrix} x_1^2 & x_3^2 & x_5^2 \\ y_1^2 & y_3^2 & y_5^2 \\ 2x_1y_1 & 2x_3y_3 & 2x_5y_5 \end{bmatrix} \cdot \begin{bmatrix} A_1 + A_4 \\ A_3 + A_6 \\ A_5 + A_2 \end{bmatrix} = \begin{bmatrix} 2D_1 \\ 2D_2 \\ 0 \end{bmatrix},$$

$$(13)\qquad M_{asym} \cdot A^{asym} \equiv \begin{bmatrix} x_1 & x_3 & x_5 \\ y_1 & y_3 & y_5 \\ x_1^3 & x_3^3 & x_5^3 \\ y_1^3 & y_3^3 & y_5^3 \end{bmatrix} \cdot \begin{bmatrix} A_1 - A_4 \\ A_3 - A_6 \\ A_5 - A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Invertibility of the matrices M_{sym} and M_{asym} (implicitly defined above) will be determined. Firstly, we apply one elementary column operation by adding the first two columns to the third one of the matrix of the asymmetric part (Eq.(13)), and deduce that

$$(14)\qquad M_3 \cdot * \equiv \begin{bmatrix} x_1 & x_3 & 0 \\ y_1 & y_3 & 0 \\ x_1^3 & x_3^3 & 3x_1x_3x_5 \\ y_1^3 & y_3^3 & 3y_1y_3y_5 \end{bmatrix} \cdot \begin{bmatrix} A_1 - A_4 - (A_5 - A_2) \\ A_3 - A_6 - (A_5 - A_2) \\ A_5 - A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

by barycentric relations, Eqs.(25,28 and 29).

Lemma 2. *The asymmetric part, A^{asym} in Eq.* (13), *has* (*unique*) *trivial solution,*

$$A^{asym} = 0, \quad \text{that is,} \quad A_j = A_{j+3}, \qquad j = 1, 2, 3.$$

Proof. Just observe by geometric argument that

(i) (x_1, y_1) and (x_3, y_3) are not colinear, and

(ii) at most one of $\{x_1, x_3, x_5, y_1, y_3, y_5\}$ can be zero.

Therefore, the 4-by-3 matrix, M_3, has (full) rank three. So is the matrix M_{asym}. □

Next, we use

$$(x_j, y_j) = d(\cos\xi_j, \sin\xi_j), \quad j = 1, 2, ..., 6, \qquad \xi_{j+3} = \xi_j + \pi, \quad j = 1, 2, 3,$$

to recast Eq.(12) as

$$d^2 \begin{bmatrix} \cos^2\xi_1 & \cos^2\xi_3 & \cos^2\xi_5 \\ \sin^2\xi_1 & \sin^2\xi_3 & \sin^2\xi_5 \\ \sin 2\xi_1 & \sin 2\xi_3 & \sin 2\xi_5 \end{bmatrix} \cdot \begin{bmatrix} A_1 + A_4 \\ A_3 + A_6 \\ A_5 + A_2 \end{bmatrix} = \begin{bmatrix} 2D_1 \\ 2D_2 \\ 0 \end{bmatrix},$$

or equivalently, by elementary row operations,

$$d^2 M_2 A^{sym} \equiv d^2 \begin{bmatrix} 1 & 1 & 1 \\ \cos 2\xi_1 & \cos 2\xi_3 & \cos 2\xi_5 \\ \sin 2\xi_1 & \sin 2\xi_3 & \sin 2\xi_5 \end{bmatrix} \begin{bmatrix} A_1 + A_4 \\ A_3 + A_6 \\ A_5 + A_2 \end{bmatrix} = \begin{bmatrix} 2D_1 + 2D_2 \\ 2D_1 - 2D_2 \\ 0 \end{bmatrix}.$$

Note the above-defined M_2 has nonzero determinant,

$$\det(M_2) = \sin(2\xi_5 - 2\xi_3) - \sin(2\xi_5 - 2\xi_1) + \sin(2\xi_3 - 2\xi_1) = \frac{-3\sqrt{3}}{2},$$

by the relations $\xi_5 - \xi_3 = \xi_3 - \xi_1 = 2\pi/3$. It is easier now to work with cofactors and obtain

$$(15) \quad M_2^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2\cos 2\xi_1 & 2\sin 2\xi_1 \\ 1 & 2\cos 2\xi_3 & 2\sin 2\xi_3 \\ 1 & 2\cos 2\xi_5 & 2\sin 2\xi_5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -2\cos 2\varphi & -2\sin 2\varphi \\ 1 & -2\cos(2\varphi - \frac{2\pi}{3}) & -2\sin(2\varphi - \frac{2\pi}{3}) \\ 1 & -2\cos(2\varphi + \frac{2\pi}{3}) & -2\sin(2\varphi + \frac{2\pi}{3}) \end{bmatrix}.$$

Consequently, by the equivalence that

$$M_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot M_{sym} \qquad \text{and} \qquad M_{sym}^{-1} = M_2^{-1} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we obtain

$$M_{sym}^{-1} = \frac{2}{3d^2} \begin{bmatrix} \frac{1}{2} + \cos 2\xi_1 & \frac{1}{2} - \cos 2\xi_1 & \sin 2\xi_1 \\ \frac{1}{2} + \cos 2\xi_3 & \frac{1}{2} - \cos 2\xi_3 & \sin 2\xi_3 \\ \frac{1}{2} + \cos 2\xi_5 & \frac{1}{2} - \cos 2\xi_5 & \sin 2\xi_5 \end{bmatrix}$$

$$= \frac{2}{3d^2} \begin{bmatrix} \frac{1}{2} - \cos 2\varphi & \frac{1}{2} + \cos 2\varphi & -\sin 2\varphi \\ \frac{1}{2} - \cos(2\varphi - \frac{2\pi}{3}) & \frac{1}{2} + \cos(2\varphi - \frac{2\pi}{3}) & -\sin(2\varphi - \frac{2\pi}{3}) \\ \frac{1}{2} - \cos(2\varphi + \frac{2\pi}{3}) & \frac{1}{2} + \cos(2\varphi + \frac{2\pi}{3}) & -\sin(2\varphi + \frac{2\pi}{3}) \end{bmatrix}.$$

We state formally

Lemma 3. *The symmetric part A^{sym} in Eq.(12) is uniquely solved in terms of D_1, D_2, d and φ. Morever, $A_0 = -(D_1 + D_2)/(2h^2)$ independent of the phase angle.*

Proof. Inversion of Eq.(12) yields

$$\begin{bmatrix} A_1 + A_4 \\ A_3 + A_6 \\ A_5 + A_2 \end{bmatrix} = \frac{2}{3d^2} \begin{bmatrix} D_1 + D_2 - 2(D_1 - D_2)\cos\, 2\varphi \\ D_1 + D_2 - 2(D_1 - D_2)\cos(2\varphi - \frac{2\pi}{3}) \\ D_1 + D_2 - 2(D_1 - D_2)\cos(2\varphi + \frac{2\pi}{3}) \end{bmatrix}, \tag{16}$$

and, by the linear barycentric identity (Eq.(25)),

$$-A_0 = \sum_{j=1}^{6} A_j = \frac{2(D_1 + D_2)}{d^2} = \frac{D_1 + D_2}{2h^2}, \tag{17}$$

as claimed. □

We conclude on *the* H7 scheme.

Theorem 1. *The ordinary hexagonal seven-point stencil is uniquely determined by (seven out of) the eight interpolants, $u \in \{1, x, y, x^2, y^2, xy, x^3$ or $y^3\}$, using (LU or) SVD method for all phase angles. The resulting stencil is symmetric and exact on all odd polynomials (by Lemma 1).*

Three typical examples of H7 schemes are given collectively in Section 4.5 with $\{R_j\}_0^6 = \{1, 0, 0, 0, 0, 0, 0\}$.

The functional approach, Eq.(5), indicates the now well-justified H7 scheme is of (at least) second-order. A *formal* proof is given in Appendix B for Hex II FVs with explicit leading error term.

4. COMPACT SEVEN-POINT SCHEME

For H7c, we note that

1. Cases 4,5,6 ($u = x^2, y^2, xy$) together yield the same linear system as with an H7 scheme. Therefore Eq.(12) and Lemma 3 apply with H7c as well.

2. Should $R^{asym} = 0$ in an H7c scheme, cases $\{x, y, x^3, y^3\}$ would yield the same Eq.(13) and then Lemma 2 would apply. However, the R-stencil is not necessarily symmetric as shown next.

4.1. Non-unique asymmetric part : R^{asym}

Lemma 4. *Associated with the linear and cubic monomials $\{x, y, x^3, y^3, x^2y, xy^2\}$, the interpolation system*

$$\begin{bmatrix} x_1 & x_3 & x_5 & 0 & 0 & 0 \\ y_1 & y_3 & y_5 & 0 & 0 & 0 \\ x_1^3 & x_3^3 & x_5^3 & -6D_1x_1 & -6D_1x_3 & -6D_1x_5 \\ y_1^3 & y_3^3 & y_5^3 & -6D_2y_1 & -6D_2y_3 & -6D_2y_5 \\ x_1^2y_1 & x_3^2y_3 & x_5^2y_5 & -2D_1y_1 & -2D_1y_3 & -2D_1y_5 \\ x_1y_1^2 & x_3y_3^2 & x_5y_5^2 & -2D_2x_1 & -2D_2x_3 & -2D_2x_5 \end{bmatrix} \begin{bmatrix} A_1 - A_4 \\ A_3 - A_6 \\ A_5 - A_2 \\ R_1 - R_4 \\ R_3 - R_6 \\ R_5 - R_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

has nonunique (nontrivial) solutions.

Proof. The elementary column-operation, adding the fourth and fifth columns to the sixth one, leads to a homogeneous system, in which the sixth column are identically zero by the linear barycentric identity. Hence the non-uniqueness of solutions, which completes the proof. □

We note the above *similarity* argument implies that $(R_1 + R_3 + R_5 - R_2 - R_4 - R_6)$ $(= 3\delta_R$ from Lemma 5 to come) is a free parameter.

Next we investigate further structures of the coupled solution

$$\{A_1^{asym}, A_3^{asym}, A_5^{asym},\ R_1^{asym}, R_3^{asym}, R_5^{asym}\}.$$

4.2. Vanishing A^{asym} and constant R^{asym}

By the linear barycentric identity, the 2-by-3 subsystem consisting of the two linear cases $(u = x, y)$ in Eq.(18 or 13) is equivalent to the following 2-by-2 system

$$\begin{bmatrix} x_1 & x_3 \\ y_1 & y_3 \end{bmatrix} \begin{bmatrix} A_1 - A_4 - (A_5 - A_2) \\ A_3 - A_6 - (A_5 - A_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which admits the unique trivial solution because the three points $\{P_0, P_1, P_3\}$ are not colinear. Therefore, $A_1 - A_4 = A_5 - A_2 = A_3 - A_6 = \delta_A$ is a constant.

By this and the cubic barycentric identities in Lemma 8, the four cubic cases in Eq.(18) simplify to

$$\begin{bmatrix} x_1 & x_3 \\ y_1 & y_3 \\ y_1 & y_3 \\ x_1 & x_3 \end{bmatrix} \begin{bmatrix} R_1 - R_4 - (R_5 - R_2) \\ R_3 - R_6 - (R_5 - R_2) \end{bmatrix} = \delta_A \begin{bmatrix} (x_1^3 + x_3^3 + x_5^3)/(6D_1) \\ (y_1^3 + y_3^3 + y_5^3)/(6D_2) \\ (x_1^2y_1 + x_3^2y_3 + x_5^2y_5)/(2D_1) \\ (x_1y_1^2 + x_3y_3^2 + x_5y_5^2)/(2D_2) \end{bmatrix} = \frac{3}{4}\,\delta_A \begin{bmatrix} \sin 3\varphi/(6D_1) \\ \cos 3\varphi/(6D_2) \\ -\cos 3\varphi/(2D_1) \\ -\sin 3\varphi/(2D_2) \end{bmatrix}. \tag{19}$$

It must be true that $\delta_A = 0$. Otherwise, the consistence in the first and the fourth rows in the above 4-by-2 linear system requires $\sin 3\varphi = 0$, while $\cos 3\varphi = 0$ by the second and third rows. Thus a contradiction. We conclude that $A^{asym} = 0$ and R^{asym} is a constant.

We summarize on the asymmetric parts of H7c stencils.

Lemma 5. *To meet the linear and cubic interpolation conditions (Eq.(18)), it is necessary and sufficient that*

$$A_1 - A_4 = A_5 - A_2 = A_3 - A_6 = 0, \tag{20}$$

$$R_1 - R_4 = R_5 - R_2 = R_3 - R_6 = \delta_R, \qquad \textit{being a constant.} \tag{21}$$

Remark 1.

1. If $\delta_R = 0$, the stencil $\{R_j\}_{j=1}^{6}$ satisfies the symmetry (Definition 1). Otherwise, it is unique up to a multiple of *period-three-rotation* in the form $\{1, 0, 1, 0, 1, 0\}$. However, interpolation conditions on degree-five monomials are generally not satisfied with such asymmetric R_j's. Although, it may have advantages in some application(s) up to further research.
2. Even if we assume the symmetry ($R_j = R_{j+3}$) and then the total stencil solution ($\{A_j, R_j\} = \{A^{sym}, R^{sym}\}$) is uniquely solvable, but depending on the choice of interpolants, as will be detailed in Remark 2.

4.3. Absolutely unique A^{sym}

Now $A^{asym} = 0$ and $A = A^{sym}$ are explicitly solved by Eqs.(16, 17). The condition of stability that all off-center weights $\{A_j\}_1^6$ be positive is easily seen.

Corollary 1. *The necessary and sufficient condition that* $A_j > 0,\ j = 1, 2, \ldots, 6,$ *is*

$$\frac{(D_1 + D_2)/2}{D_1 - D_2} > \max\left(\cos(2\varphi),\ \cos(2\varphi \pm 2\pi/3)\right), \qquad \text{if } D_1 > D_2,$$

$$\frac{(D_1 + D_2)/2}{D_2 - D_1} > \max\left(\cos(2(\pi/2 - \varphi)),\ \cos(2(\pi/2 - \varphi) \pm 2\pi/3)\right), \qquad \text{if } D_2 > D_1.$$

Also,

Corollary 2. *The necessary and sufficient condition that* $A_j > 0,\ j = 1, 2, ..., 6$ *, independent of the phase angle* (φ)*, is*

$$\frac{1}{3} < \frac{D_2}{D_1} < 3.$$

Proof. Besides the trivial case $D_1 = D_2$, we observe that

$$\text{If } D_1 > D_2, \quad \text{then} \quad \frac{(D_1 + D_2)/2}{D_1 - D_2} > 1 \iff \frac{1}{3} < \frac{D_2}{D_1} < 1.$$

$$\text{If } D_2 > D_1, \quad \text{then} \quad \frac{(D_1 + D_2)/2}{D_2 - D_1} > 1 \iff 1 < \frac{D_2}{D_1} < 3.$$

This finishes the proof.

4.4. Up-to-interpolants unique R^{sym}

The three cases, $u \in \{x^4, y^4, x^3y\}$ in Table 3, lead to (with $D = D_1$)

$$\begin{bmatrix} 12D_1 & 0 & 0 \\ 0 & 12D_2 & 0 \\ 0 & 0 & 3D \end{bmatrix} M_{sym} \begin{bmatrix} R_1 + R_4 \\ R_3 + R_6 \\ R_5 + R_2 \end{bmatrix} \tag{22}$$
$$= d^4 \begin{bmatrix} \cos^4 \xi_1 & \cos^4 \xi_3 & \cos^4 \xi_5 \\ \sin^4 \xi_1 & \sin^4 \xi_3 & \sin^4 \xi_5 \\ \cos^3 \xi_1 \sin \xi_1 & \cos^3 \xi_3 \sin \xi_3 & \cos^3 \xi_5 \sin \xi_5 \end{bmatrix} \begin{bmatrix} A_1 + A_4 \\ A_3 + A_6 \\ A_5 + A_2 \end{bmatrix}.$$

The right-hand side above (with action on A^{sym}) being straight, R^{sym} is thus uniquely solvable.

The alternative approach, using $\{x^4, y^4, xy^3\}$ instead, leads to the same left-hand side in Eq.(22) with $D = D_2$ and consistent changes in the third matrix row on the right.

We state formally.

Lemma 6. *Depending on using either $\{x^4, y^4, x^3y\}$ or $\{x^4, y^4, xy^3\}$, specific R^{sym} is solved uniquely in terms of A^{sym}.*

Remark 2. Let $R = R^{sym}$ from Eq.(22) and $R^{asym} = 0 = \delta_R$ in Eqs.(18 and 21). The following statements are true.

1. The so-constructed H7c approximation is nearly fourth-order in the sense that it is exact on monomials of degrees less than six and mismatches at most the two (shifted) monomials, xy^3 and x^2y^2.
2. Should we use the monomials $\{x^4, y^4, xy^3\}$ in solving for R_j's , different R^{sym} may be obtained, mismatching x^2y^2 and (probably) x^3y.
3. In case of Hex I (II) FVs, the stencils determined by using xy^3 (x^3y) catches x^3y (xy^3) as well, and mismatches only the x^2y^2 term. Thus almost a fourth-order method, and indeed a fourth-order one if $D_1 = D_2$.
4. However, the next choice (using x^4, y^4, x^2y^2) is not suggested, with the left-hand side in Eq.(22) being a rank-deficient square system :

$$\begin{bmatrix} 12D_1 & 0 & 0 \\ 0 & 12D_2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1^2 & x_3^2 & x_5^2 \\ y_1^2 & y_3^2 & y_5^2 \\ D_1y_1^2 + D_2x_1^2 & D_1y_3^2 + D_2x_3^2 & D_1y_5^2 + D_2x_5^2 \end{bmatrix} \begin{bmatrix} R_1 + R_4 \\ R_3 + R_6 \\ R_5 + R_2 \end{bmatrix}.$$

4.5. Summary on H7c

Theorem 2. *The hexagonal compact seven-point stencils $\{A_j\}_0^6$ and $\{R_j\}_0^6$ can be determined by the normalization constraint (Eq.(6)) and the 13 interpolants,*

$$u \in \{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^4, y^4, x^3y\}.$$

If the monomial x^3y is replaced by xy^3, different solution pairs of stencils are obtained. In either case, the unique stencil $\{A_j\}$ is always symmetric while $\{R_j\}$ assumed so. Consequently, the resulting H7c scheme is exact on all odd polynomials.

To solve Eqs. (8) numerically at one strike, the symmetry ($R^{asym} = 0$) may or may not need special arrangement, depending on how the SVD method is implemented.

We note three special cases of Eqs.(16, 22).

EXAMPLE 1. Hex I, phase angle $\varphi = 0$ ($\xi_1 = \pi/2,\ \xi_3 = 7\pi/6,\ \xi_5 = 11\pi/6$),

$$\begin{aligned} &(A_1, A_3, A_5) = (A_4, A_6, A_2) = \frac{1}{3d^2}\big(\,3D_2 - D_1,\ 2D_1,\ 2D_1\big), \quad A_0 = \frac{-6(D_1 + D_2)}{3d^2}, \\ &(R_1, R_3, R_5) = (R_4, R_6, R_2) = \left(\frac{3 - D_1/D_2}{48}, \frac{1}{24}, \frac{1}{24}\right), \qquad R_0 = \frac{17 + D_1/D_2}{24}. \end{aligned} \tag{23}$$

For stability, all off-center weights are nonnegative if $D_2/D_1 \geq 1/3$.

EXAMPLE 2. Hex II, phase angle $\varphi = -\pi/6$ ($\xi_1 = \pi/3,\ \xi_3 = \pi,\ \xi_5 = 5\pi/3$),

$$\begin{aligned} &(A_1, A_3, A_5) = (A_4, A_6, A_2) = \frac{1}{3d^2}\big(\,2D_2,\ 3D_1 - D_2,\ 2D_2\,\big), \quad A_0 = \frac{-6(D_1 + D_2)}{3d^2}, \\ &(R_1, R_3, R_5) = (R_4, R_6, R_2) = \left(\frac{1}{24}, \frac{3 - D_2/D_1}{48}, \frac{1}{24}\right), \qquad R_0 = \frac{17 + D_2/D_1}{24}. \end{aligned} \tag{24}$$

All off-center weights are nonnegative if $D_2/D_1 \leq 3$.

EXAMPLE 3. The standard (fourth-order) H7c scheme ([**2**]) can be re-discovered, with $D_1 = D_2 = 1$ and for any phase angle φ , that

$$A = \frac{1}{6h^2}(-6, 1, 1, 1, 1, 1, 1) \quad \text{and} \quad R = \frac{1}{24}(18, 1, 1, 1, 1, 1, 1).$$

In the current example, either x^3y or xy^3 can be used in a similar construction of Eq.(22), and interpolation conditions (Eq.(8)) are satisfied for all polynomials of degrees less than six ($m + n < 6$).

Actually, the second example above can be obtained from the first one, as follows.

Lemma 7 (Reflection Principle)**.** *The two configurations, Hex I and Hex II together with the diagonal-form Laplacian, are convertible from each other by applying reflection with respect to the main diagonal in the xy-plane, and therefore interchanging the two symbol lists* (*Eq.* (3))

$$\big\{x, y, D_1, D_2, N, NW, SW, S, SE, NE\big\} \quad \text{and} \quad \big\{y, x, D_2, D_1, E, SE, SW, W, NW, NE\big\}.$$

With a general phase angle, the reflection interchanges

$$\big\{\ (\varphi),\ \{P_j\}_1^6\ \big\} \qquad \text{and} \qquad \big\{\ (\pi/2 - \varphi),\ \{P_{6-j\%6}\}_1^6\ \big\}.$$

Here $\big\{P_{6-j\%6}\big\}_1^6$ *refers to the outcome of the order-*2 *permutation* $\binom{1\,2\,3\,4\,5\,6}{5\,4\,3\,2\,1\,6}$ *of the indices.*

Proof. All correspondence in the geometry are based on the relation

$$\left(\sin\left(\psi+\frac{j\pi}{3}\right),\cos\left(\psi+\frac{j\pi}{3}\right)\right)=\left(\cos\left(\frac{\pi}{2}-\left(\psi+\frac{j\pi}{3}\right)\right),\ \sin\left(\frac{\pi}{2}-\left(\psi+\frac{j\pi}{3}\right)\right)\right)$$
$$=\left(\cos\left(\frac{\pi}{2}-\psi+\frac{(6-j)\pi}{3}\right),\ \sin\left(\frac{\pi}{2}-\psi+\frac{(6-j)\pi}{3}\right)\right).$$

This ends the proof.

5. CELL-VERTEX HEXAGONAL FV METHOD FAILS

In a cell-vertex (other than cell-center) FV approach, we *try* to estabilish the relation,

$$D_1u_{xx}+D_2u_{yy}\approx\sum_{j=1}^{6}B_ju_{V_j},$$

using the six vertices of the *cell* (FV),

$$V_j=P_0+r(\cos\theta_j,\sin\theta_j),\quad \theta_j=\varphi+\frac{j\pi}{3},\quad j=1,2,...,6.$$

The first interpolant ($u=1,m=n=0$) requires

$$\sum_{j=1}^{6}B_j=0.$$

All the previous discussions about an H7 scheme involving only $\{A_j\}_1^6$ are applicable to $\{B_j\}_1^6$, with neighbor centers $\{P_j\}$ replaced by local vertices $\{V_j\}$, that is, with d replaced by r and φ by $(\varphi-\pi/6)$. Then the relation,

$$\sum_{1}^{6}B_j=\frac{2(D_1+D_2)}{r^2}\neq 0,$$

a mimic of Eq.(17), denies the cell-vertex ordinary hexagonal method as formulated. A similar compact hexagonal scheme fails either.

6. CONCLUSIONS

Cell-centered hexagonal finite volume method were confirmed effective in standard Poisson problems and also in time-dependent problems such as to exhibit successfully linear and spiral waves ([**2**]). The basic scheme is extended here to diagonal-form Laplacian and investigated by even-odd decompositions of the stencils. Complete and explicit solutions are obtained for both ordinary and compact hexagonal seven-point methods, with the latter being of almost fourth-order on type I and II hexagons, mismatching only the monomial x^2y^2 among all monomials of degrees at most five, and nearly fourth-order on hexagons in general phase, mismatching two degree-four monomials x^2y^2 and x^3y (or xy^3), as explained in Remark 2. The discussion after Eq.(4) may lead to staggered grid approach to some flow problems. The application to wide range of numerical wave propagation and image science is potentially of much value. We think this applies in many *open field* type applications in two-dimensional irregular domains.

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Appendix A. SOME BARYCENTRIC RELATIONS

Let φ be the phase angle, $d = 1$ and $P_0 = (0, 0)$ in our setup, Eq.(2).

Lemma 8. *With* $\psi(= \varphi + \pi/6)$ *a start angle, consider*

$$(x_j, y_j) = (\cos\xi_j, \sin\xi_j), \quad \xi_j = \psi + \frac{j\pi}{3}, \quad j = 1, 3, 5.$$

The following barycentric identities are valid.

$$x_1 + x_3 + x_5 = y_1 + y_3 + y_5 = 0, \tag{25}$$

$$x_1^2 + x_3^2 + x_5^2 = y_1^2 + y_3^2 + y_5^2 = \frac{3}{2}, \tag{26}$$

$$x_1^4 + x_3^4 + x_5^4 = y_1^4 + y_3^4 + y_5^4 = \frac{9}{8}, \tag{27}$$

$$x_1^3 + x_3^3 + x_5^3 = 3x_1x_3x_5 = \frac{-3}{4}\cos 3\psi = \frac{3}{4}\sin 3\varphi, \tag{28}$$

$$y_1^3 + y_3^3 + y_5^3 = 3y_1y_3y_5 = \frac{3}{4}\sin 3\psi = \frac{3}{4}\cos 3\varphi, \tag{29}$$

$$x_1^2y_1 + x_3^2y_3 + x_5^2y_5 = \frac{-3}{4}\sin 3\psi = \frac{-3}{4}\cos 3\varphi, \tag{30}$$

$$x_1y_1^2 + x_3y_3^2 + x_5y_5^2 = \frac{3}{4}\cos 3\psi = \frac{-3}{4}\sin 3\varphi. \tag{31}$$

Proof. The linear case follows by the geometric argument that

$$P_1 + P_3 + P_5 = 3P_0 = (0, 0),$$

while the two even degree cases can be derived in routine ways ([**2**]). As for the four cubic cases, the first two are obtained by observing that

$$\begin{aligned} x_1^3 + x_3^3 - (x_1 + x_3)^3 &= -3x_1x_3(x_1 + x_3) &= 3x_1x_3x_5, \\ y_1^3 + y_3^3 - (y_1 + y_3)^3 &= -3y_1y_3(y_1 + y_3) &= 3y_1y_3y_5, \end{aligned}$$

and using two general formulas

$$4\cos\alpha\cos\beta\cos\gamma = \cos(\alpha - \beta + \gamma) + \cos(\alpha + \beta - \gamma) + \cos(-\alpha + \beta + \gamma) + \cos(\alpha + \beta + \gamma),$$
$$4\sin\alpha\sin\beta\sin\gamma = \sin(\alpha - \beta + \gamma) + \sin(\alpha + \beta - \gamma) + \sin(-\alpha + \beta + \gamma) - \sin(\alpha + \beta + \gamma).$$

Finally, the last two cases can be proved by using

$$4\sin\alpha\cos\beta\cos\gamma = \sin(\alpha-\beta+\gamma)+\sin(\alpha+\beta-\gamma)-\sin(-\alpha+\beta+\gamma)+\sin(\alpha+\beta+\gamma),$$
$$4\sin\alpha\sin\beta\cos\gamma = \cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta-\gamma)+\cos(-\alpha+\beta+\gamma)-\cos(\alpha+\beta+\gamma),$$

or alternatively, by observing

$$(x_1^3+x_3^3+x_5^3)+(x_1y_1^2+x_3y_3^2+x_5y_5^2) = x_1(x_1^2+y_1^2)+x_3(x_3^2+y_3^2)+x_5(x_5^2+y_5^2)=0,$$
$$(y_1^3+y_3^3+y_5^3)+(y_1x_1^2+y_3x_3^2+y_5x_5^2) = y_1(x_1^2+y_1^2)+y_3(x_3^2+y_3^2)+y_5(x_5^2+y_5^2)=0.$$

This ends the proof. □

We note *complex* argument could be used to replace part of the argument above. A reference to the four general trig-formulas involving product of three factors is ([**6**]).

Remark 3.

1. By Lemma 7, reflection of the configuration wrt the main diagonal in the xy-plane results in interchanging the two symbol lists,

   $$\{\ x_1,x_3,x_5,\ \ y_1,y_3,y_5,\ \ (\psi)\ \} \quad \text{and} \quad \{\ y_5,y_3,y_1,\ \ x_5,x_3,x_1,\ \ (\pi/2-\psi)\ \}.$$

   Only half of the claimed identities need proofs, except for Eqs.(26,27).
2. All the barycentric identities remain valid if the indices $\{1,3,5\}$ are replaced by $\{2,4,6\}$. The latter corresponds to a configurarion with φ replaced by $\varphi+\frac{\pi}{3}$, while still using $j=1,3,5$.

Appendix B. H7 SCHEME WITH ERROR ESTIMATE : TYPE II HEXAGONS

Assuming appropriate smoothness of the function u and using the coordinates (Eq.(2)) and abbreviations (Eq.(3)) at a Hex II FV, truncated Taylor expansions yield,

$$\begin{aligned} u_{NE}+u_{SE} &\approx 2u_P + 2\left(\frac{d}{2}u_x+\frac{d^2}{2}\Big(\frac{1}{4}\,u_{xx}+\frac{3}{4}\,u_{yy}\Big)+\frac{d^3}{6}\Big(\frac{1}{8}\,u_{xxx}+\frac{9}{8}\,u_{xyy}\Big)\right.\\ &\left.\quad+\frac{d^4}{24}\Big(\frac{1}{16}\,u_{xxxx}+\frac{18}{16}\,u_{xxyy}+\frac{9}{16}\,u_{yyyy}\Big)\right),\\ u_{NW}+u_{SW} &\approx 2u_P + 2\left(\frac{-d}{2}u_x+\frac{d^2}{2}\Big(\frac{1}{4}\,u_{xx}+\frac{3}{4}\,u_{yy})-\frac{d^3}{6}\Big(\frac{1}{8}\,u_{xxx}+\frac{9}{8}\,u_{xyy})\right.\\ &\left.\quad+\frac{d^4}{24}\Big(\frac{1}{16}\,u_{xxxx}+\frac{18}{16}\,u_{xxyy}+\frac{9}{16}\,u_{yyyy}\Big)\right),\end{aligned}$$

and

$$\begin{aligned} u_{NE}+u_{NW}+u_{SE}+u_{SW}-4u_P &= d^2\left(\frac{1}{2}\,u_{xx}+\frac{3}{2}\,u_{yy}\right)+\\ &+d^4\left(\frac{1}{96}\,u_{xxxx}+\frac{18}{96}\,u_{xxyy}+\frac{9}{96}\,u_{yyyy}\right)+\mathcal{O}(d^6).\end{aligned}$$

Also

$$u_E + u_W - 2u_P = d^2 u_{xx} + \frac{d^4}{12} u_{xxxx} + \mathcal{O}(d^6).$$

A linear combination of the last two equations yields a second-order seven-point scheme for the diagonal-form Laplacian with an error estimate,

$$\frac{2D_2(u_{NE} + u_{NW} + u_{SE} + u_{SW}) + (3D_1 - D_2)(u_E + u_W) - (6D_1 + 6D_2)u_P}{3d^2}$$
$$= D_1 u_{xx} + D_2 u_{yy} + \frac{d^2}{48}\Big((4D_1 - D_2)u_{xxxx} + 6D_2 u_{xxyy} + 3D_2 u_{yyyy}\Big) + \mathcal{O}(d^4),$$

for type II hexagons. □

We note by reflection (Lemma 7),

$$\frac{2D_1(u_{NE} + u_{NW} + u_{SE} + u_{SW}) + (3D_2 - D_1)(u_N + u_S) - 6(D_1 + D_2)u_P}{3d^2}$$
$$= D_1 u_{xx} + D_2 u_{yy} + \frac{d^2}{48}\Big(3D_1 u_{xxxx} + 6D_1 u_{xxyy} + (4D_2 - D_1)u_{yyyy}\Big) + \mathcal{O}(d^4).$$

for type I hexagons.

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