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POSITIVE SOLUTIONS TO ADVANCED FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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We study the existence of positive solutions for a class of higher order fractional differential equations with advanced arguments and boundary value problems involving Stieltjes integral conditions. The fixed point theorem due to Avery–Peterson is used to obtain sufficient conditions for the existence of multiple positive solutions. Certain of our results improve on recent work in the literature.

1. INTRODUCTION

Fractional differential equations (FDEs) can describe many phenomena in various fields of science and engineering. FDEs have been discussed in many papers, for example, see [**3**], [**5**], [**9**]–[**18**]. Many authors have studied the existence of positive solutions by using corresponding fixed point theorems in cones, for example, see [**3**], [**12**], [**15**], [**17**], [**18**].

Put $J=[0,1],\ J_0=(0,1)$. In this paper, we are interested in the existence of multiple positive solutions to boundary value problem:

$$(1)\qquad \begin{cases} D^q x(t)+f(t,x(\alpha(t)))=0, \quad t\in J_0,\ n-1<q\le n,\ n\ge 3,\\ x^{(i)}(0)=0, \quad 0\le i\le n-2,\\ \left[D^k x(t)\right]_{t=1}=\lambda[x], \quad k \quad \text{is a fixed number and} \quad k\in[1,n-2], \end{cases}$$

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for $n \geq 3$, where λ denotes a linear functional on $C(J)$ given by

$$\lambda[x] = \int_0^1 x(t) \mathrm{d}\Lambda(t)$$

involving a Stieltjes integral with a suitable function Λ of bounded variation. Linear functional $\lambda[x]$ covers the multi–point Boundary Conditions (BCs) and also integral BCs, see Section 4. It is important to indicate that it is not assumed that $\lambda[x]$ is positive to all positive x. The measure $\mathrm{d}\Lambda$ can be a signed measure (see Remark 3). It is important to indicate that the situation with a signed measure has been discussed, for example, in [**13**], [**4**], [**6**], [**7**] for second or third-order ordinary differential equations. A unified approach for higher order problems with nonlocal conditions and signed measure has been given in [**14**]. A physical application to heat–flow problems of second–order nonlocal boundary value problems with deviated arguments has been studied in [**2**].

Some authors studied higher order fractional differential equations (FDEs) with different BCs, for example,

$$\begin{aligned}
&x^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \quad x(1) = 0,\\
&x^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \quad x(1) = \lambda[x],\\
&x^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \quad x^{(n-2)}(1) = 0,\\
&x^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \quad \left[D^k x(t)\right]_{t=1} = 0, \quad k \in [1, n-2],\\
&x^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \quad \left[D^k x(t)\right]_{t=1} = \sum_{i=1}^{m} \beta_i x(\xi_i), \quad k \in [1, n-2],
\end{aligned}$$

see [**3**], [**12**], [**15**], [**17**], [**18**], see also [**8**]. In the mentioned papers, FDEs without deviating arguments have been discussed using the Krasnoselskii's fixed point theorem in a cone or a monotone iterative method to obtain the existence of positive solutions.

Motivated by [**3**], [**12**], [**15**], [**17**], [**18**] and [**13**], [**14**], in this paper, we apply the Avery–Peterson fixed point theorem to obtain sufficient conditions for the existence of positive solutions to problem (1). In this paper we improve certain results obtained in papers [**3**], [**12**], [**17**]. Note that the existence results have been obtained for quite general problems of type (1) with advanced arguments α. The measure $d\Lambda$ in BCs of (1) can change the sign, see Remark 3. In Section 4, special cases of functional $\lambda[x]$ have been discussed.

2. GREEN'S FUNCTION PROPERTIES

First we introduce the following assumptions:

$H_1:$ $f \in C(J \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$, $\alpha \in C(J, J)$, $\alpha(t) \geq t$ with $\mathbb{R}_+ = [0, \infty)$,

$H_2: \ 0 \le \int_0^1 t^{q-1} \mathrm{d}\Lambda(t) < \frac{\Gamma(q)}{\Gamma(q-k)}, \quad n-1 < q \le n, \ k \in [1, n-2], \ n \ge 3,$

$H_3: \ \int_0^1 \mathrm{d}\Lambda(t) \ge 0.$

By $D^q x$, we denote the Riemann-Liouville fractional derivative of order $q > 0$, and by $I^q x$, the Riemann-Liouville fractional integral of order $q > 0$, see [**9**],[**11**], so

$$\begin{aligned} D^q x(t) &= \frac{1}{\Gamma(n-q)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{-q+n-1} x(s) \mathrm{d}s, \quad n = [q]+1, \ q > 0, \ t < 1, \\ D^n x(t) &= y^{(n)}(t), \quad n \in \{1, 2, 3, \cdots\}, \\ I^q x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s) \mathrm{d}s, \quad q > 0, \ t < 1, \end{aligned}$$

where $[q]$ means the integer part of q.

Consider the following boundary value problem:

$$\text{(2)} \qquad \begin{cases} D^q u(t) + y(t) = 0, \quad t \in J_0, \ n-1 < q \le n, \ n \ge 3, \\ u^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ \left[D^k u(t)\right]_{t=1} = \lambda[u], \quad k \quad \text{is a fixed number and} \quad k \in [1, n-2], \end{cases}$$

We require the following assumption:

$H_0:$ Λ is a function of bounded variation and

$$\Delta_1 \equiv \Gamma(q) - \Gamma(q-k)A \ne 0, \quad \Delta = \frac{\Delta_1}{\Gamma(q-k)},$$
$$A = \int_0^1 t^{q-1} d\Lambda(t), \quad \mathcal{G}(s) = \int_0^1 G_1(t,s) \mathrm{d}\Lambda(t)$$
$$G_1(t,s) = \frac{1}{\Gamma(q)} \begin{cases} t^{q-1}(1-s)^{q-k-1} - (t-s)^{q-1}, & \text{if} \quad s \le t, \\ t^{q-1}(1-s)^{q-k-1}, & \text{if} \quad t \le s. \end{cases}$$

Lemma 1. *Assume that Assumption H_0 holds. Let $y \in L(J_0, I\!R)$. Then, problem (2) has the unique solution given by the following formula*

$$u(t) = \int_0^1 G_q(t,s) y(s) \mathrm{d}s,$$

where

$$G_q(t,s) = G_1(t,s) + G_2(t,s), \quad G_2(t,s) = \frac{\mathcal{G}(s)}{\Delta} t^{q-1}.$$

Proof. The general solution of (2) is given by

$$u(t) = -I^q y(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n}.$$

Indeed, $c_2 = c_3 = \cdots = c_n = 0$ in view of conditions $u^{(i)}(0) = 0,\ i = 0, 1, \ldots, n-2$, so

$$u(t) = -I^q y(t) + c_1 t^{q-1}. \tag{3}$$

Hence, in view of the property $D^k I^q = I^{q-k}$,

$$\begin{aligned} D^k u(t) &= -D^k I^q y(t) + c_1 D^k[t^{q-1}] \\ &= -\frac{1}{\Gamma(q-k)} \int_0^t (t-s)^{q-k-1} y(s) \mathrm{d}s + c_1 \frac{\Gamma(q)}{\Gamma(q-k)} t^{q-k-1}. \end{aligned}$$

This and condition $\left[D^k u(t)\right]_{t=1} = \lambda[u]$ give

$$-\frac{1}{\Gamma(q-k)} \int_0^1 (1-s)^{q-k-1} y(s) \mathrm{d}s + c_1 \frac{\Gamma(q)}{\Gamma(q-k)} = \lambda[u].$$

Finding c_1 and substituting in (3) we obtain

$$u(t) = t^{q-1} \frac{\Gamma(q-k)}{\Gamma(q)} \lambda[u] + \int_0^1 G_1(t,s) y(s) \mathrm{d}s. \tag{4}$$

In the next step, we have to eliminate $\lambda[u]$ from (4). If u is a solution of (4), then

$$\lambda[u] = \frac{\Gamma(q)}{\Delta_1} \int_0^1 \mathcal{G}(s) y(s) \mathrm{d}s.$$

Substituting it to formula (4) we finally get the assertion of this lemma.

Remark 1. Note that G_q is the Green function of problem (1).

Lemma 2. *Function G_1 from Assumption H_0 has the following property:*

$$t^{q-1}\Phi_1(s) \le G_1(t,s) \le \Phi_1(s), \quad t, s \in J,$$

where

$$\Phi_1(s) = \frac{1}{\Gamma(q)} (1-s)^{q-k-1} [1 - (1-s)^k].$$

Proof. Let $s \le t$. In view of $q > 2,\ q-k-1 \le q-2,\ t-s \le t(1-s)$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \Gamma(q) G_1(t,s) &= (q-1) \left[t^{q-2}(1-s)^{q-k-1} - (t-s)^{q-2}\right] \\ &\ge (q-1) t^{q-2} \left[(1-s)^{q-k-1} - (1-s)^{q-2}\right] \ge 0, \end{aligned}$$

so

$$\begin{aligned} \Gamma(q) G_1(t,s) &\le (1-s)^{q-k-1} - (1-s)^{q-1} \\ &= (1-s)^{q-k-1} \left[1 - (1-s)^k\right] = \Gamma(q) \Phi_1(s). \end{aligned}$$

Moreover,

$$\begin{aligned}\Gamma(q)G_1(t,s) &= t^{q-1}(1-s)^{q-k-1} - (t-s)^{q-1}\\ &= t^k[t(1-s)]^{q-k-1} - (t-s)^k(t-s)^{q-k-1}\\ &\geq [t(1-s)]^{q-k-1}[t^k-(t-s)^k] \geq [t(1-s)]^{q-k-1}[t^k-(t(1-s))^k]\\ &= t^{q-1}(1-s)^{q-k-1}\left[1-(1-s)^k\right] = t^{q-1}\Gamma(q)\Phi_1(s).\end{aligned}$$

Now, we consider the case when $t \leq s$. Indeed, $\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(q)G_1(t,s) \geq 0$, so

$$\Gamma(q)G_1(t,s) \leq s^{q-1}(1-s)^{q-k-1} \leq s(1-s)^{q-k-1} \leq \Gamma(q)\Phi_1(s)$$

because $s = 1-(1-s) \leq 1-(1-s)^k$.

Moreover,

$$\begin{aligned}\Gamma(q)G_1(t,s) &= t^{q-1}(1-s)^{q-k-1} \geq t^{q-1}(1-s)^{q-k-1}\left[1-(1-s)^k\right]\\ &= t^{q-1}\Gamma(q)\Phi_1(s)\end{aligned}$$

because $1 \geq 1-(1-s)^k$. The proof is complete.

Remark 2. Let $\Delta > 0,\ \mathcal{G}(s) \geq 0,\ s \in [0,1]$. In view of Lemma 2 and the definition of G_q, we have the estimation

$$t^{q-1}\Phi(s) \leq G_q(t,s) \leq \Phi_1(s) + \frac{1}{\Delta}\,\mathcal{G}(s) \equiv \Phi(s), \quad t,s \in J.$$

Define the operator T by

$$Tu(t) = \int_0^1 G_q(t,s)Fu(s)\mathrm{d}s \quad \text{with} \quad Fu(t) = f(t,u(\alpha(t))).$$

Take $0 < \eta < 1$ and put $\rho = \eta^{q-1}$. Let $E = C(J,\mathbb{R})$ with the norm $\|u\|$. Define the set $K \subset E$ by

$$K = \{u \in E:\ u(t) \geq 0,\ t \in J,\ \min_{[\eta,1]} u(t) \geq \rho\|u\|,\ \ \lambda[u] \geq 0\}.$$

The set K is a cone, see Definition 1.

Lemma 3. *Let Assumptions H_1, H_2, H_3 hold. Moreover, we assume that Assumption H_4 holds with*

H_4 : Λ *is of bounded variation and* $\mathcal{G}(s) \geq 0$, *where* $A, \Delta, \mathcal{G}$ *are defined as in Assumption* H_0.

Then $T : K \to K$ and T is completely continuous.

Proof. Indeed, $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$. Problem (1) has a solution u if and only if u solves the operator equation $u = Tu$. Assumptions H_1–H_4 and the positivity of the Green's function G_q prove that $Tu(t) \geq 0$.

Next, in view of Remark 2, we obtain

$$\|Tu\| = \max_{t\in J} \int_0^1 G_q(t,s)Fu(s)\mathrm{d}s \le \int_0^1 \Phi(s)Fu(s)\mathrm{d}s,$$

$$\min_{[\eta,1]} Tu(t) = \min_{[\eta,1]} \int_0^1 G_q(t,s)Fu(s)\mathrm{d}s \ge \rho\|Tu\|.$$

Indeed,

$$\lambda[Tu] = \int_0^1 \left(\int_0^1 G_q(t,s)Fu(s)\mathrm{d}s\right)\mathrm{d}\Lambda(t) = \left(1+\frac{A}{\Delta}\right)\int_0^1 \mathcal{G}(s)Fu(s)\mathrm{d}s \ge 0.$$

This proves that $T : K \to K$.

Note that

$$Tu(t) = t^{q-1}\int_0^1 \left[\frac{1}{\Gamma(q)}(1-s)^{q-k-1} + \frac{\mathcal{G}(s)}{\Delta}\right]Fu(s)\mathrm{d}s - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}Fu(s)\mathrm{d}s.$$

Now, a standard argument, which we omit, shows that T is equicontinuous and bounded, so the Arzela-Ascoli theorem may be applied to deduce the continuity of T. This ends the proof.

Remark 3. Take $\mathrm{d}\Lambda(t) = (at-1)\mathrm{d}t,\ a > 1$. Note that the measure changes the sign. Then

$$A = \int_0^1 t^{q-1}(at-1)\mathrm{d}t = \frac{q(a-1)-1}{q(q+1)}, \quad \int_0^1 \mathrm{d}\Lambda(t) = \frac{a-2}{2}.$$

Note that Assumptions H_2, H_3 hold if

$$2 \le a < 1 + \frac{1}{q} + (q+1)\frac{\Gamma(q)}{\Gamma(q-k)}.$$

For example, if $q = \frac{5}{2}$, then $k = 1$ and $2 \le a < \frac{133}{20}$.

3. POSITIVE SOLUTIONS TO PROBLEM (1)

First, we present the necessary definitions from the theory of cones in Banach spaces.

Definition 1. *Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is said to be a cone provided that*

(i) *$ku \in P$ for all $u \in P$ and all $k \ge 0$, and*

(ii) *$u, -u \in P$ implies $u = 0$.*

Note that every cone $P \subset E$ induces an ordering in E given by $x \le y$ if $y - x \in P$.

Definition 2. *A map Φ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\Phi : P \to \mathbb{R}_+$ is continuous and*

$$\Phi(tx + (1-t)y) \geq t\Phi(x) + (1-t)\Phi(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Similarly, we say the map φ is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\varphi : P \to \mathbb{R}_+$ is continuous and

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 3. *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

Let φ and Θ be nonnegative continuous convex functionals on P, let Ω be a nonnegative continuous concave functional on P, and let Ψ be a nonnegative continuous functional on P. Then, for positive numbers a, b, c, d, we define the following sets:

$$\begin{aligned}
&P(\varphi, d) = \{x \in P : \ \varphi(x) < d\},\\
&P(\varphi, \Omega, b, d) = \{x \in P : \ b \leq \Omega(x), \ \varphi(x) \leq d\},\\
&P(\varphi, \Theta, \Omega, b, c, d) = \{x \in P : \ b \leq \Omega(x), \ \Theta(x) \leq c, \ \varphi(x) \leq d\},\\
&R(\varphi, \Psi, a, d) = \{x \in P : \ a \leq \Psi(x), \ \varphi(x) \leq d\}.
\end{aligned}$$

We will use the following fixed point theorem of Avery and Peterson to establish multiple positive solutions to problem (1).

Theorem 1 (see [**1**])**.** *Let P be a cone in a real Banach space E. Let φ and Θ be nonnegative continuous convex functionals on P, let Ω be a nonnegative continuous concave functional on P, and let Ψ be a nonnegative continuous functional on P satisfying $\Psi(kx) \leq k\Psi(x)$ for $0 \leq k \leq 1$, such that for some positive numbers $\bar{M}$ and d,*

$$\Omega(x) \leq \Psi(x) \quad \text{and} \quad \|x\| \leq \bar{M}\varphi(x)$$

for all $x \in \overline{P(\varphi, d)}$. Suppose

$$T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$$

is completely continuous and there exist positive numbers a, b, c with $a < b$, such that

(S_1) : $\{x \in P(\varphi, \Theta, \Omega, b, c, d) : \Omega(x) > b\} \neq 0$ *and* $\Omega(Tx) > b$ *for* $x \in P(\varphi, \Theta, \Omega, b, c, d)$;

(S_2) : $\Omega(Tx) > b$ *for* $x \in P(\varphi, \Omega, b, d)$ *with* $\Theta(Tx) > c$,

$(S_3):$ $0 \notin R(\varphi, \Psi, a, d)$ *and* $\Psi(Tx) < a$ *for* $x \in R(\varphi, \Psi, a, d)$ *with* $\Psi(x) = a$.

Then, T *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\varphi, d)}$, *such that*

$$\varphi(x_i) \le d, \quad \textit{for} \quad i = 1, 2, 3,$$

$$b < \Omega(x_1), \quad a < \Psi(x_2), \quad \textit{with} \quad \Omega(x_2) < b \quad \textit{and} \quad \Psi(x_3) < a.$$

We apply Theorem 1 with the cone K instead of P and let $\bar{P}_r = \{x \in K : \|x\| \le r\}$. Now, we define the nonnegative continuous concave functional Ω on K by

$$\Omega(x) = \min_{[\eta,1]} |x(t)|.$$

Note that $\Omega(x) \le \|x\|$. Put $\Psi(x) = \Theta(x) = \varphi(x) = \|x\|$.

Now, we can formulate the main result of this section giving sufficient conditions under which problem (1) has positive solutions.

Theorem 2. *Let Assumptions* H_1*–*H_4 *hold. In addition, we assume that there exist positive constants* $a, b, c, d,$ $a < b$ *and such that*

$$\mu > \int_0^1 \Phi(s)\mathrm{d}s, \quad 0 < L < \rho \int_0^1 \Phi(s)\mathrm{d}s$$

with Φ *defined as in Remark* 2, *and*

$(A_1):$ $f(t,u) \le \dfrac{d}{\mu}$ *for* $(t,u) \in J \times [0,d]$,

$(A_2):$ $f(t,u) \ge \dfrac{b}{L}$ *for* $(t,u) \in [\eta,1] \times \left[b, \dfrac{b}{\rho}\right]$,

$(A_3):$ $f(t,u,v) \le \dfrac{a}{\mu}$ *for* $(t,u) \in J \times [0,a]$.

Then, problem (1) *has at least three positive solutions* x_1, x_2, x_3 *satisfying* $\|x_i\| \le d,$ $i = 1, 2, 3,$

$$b \le \Omega(x_1), \quad a < \|x_2\| \quad \textit{with} \quad \Omega(x_2) < b \quad \textit{and} \quad \|x_3\| < a.$$

Proof. Let $x \in \overline{P(\varphi, d)}$. Assumption (A_1) implies $f(t, x(\alpha(t))) \le \dfrac{d}{\mu}$. By Remark 2,

$$\varphi(Tx) = \max_{[0,1]} |(Tx)(t)| \le \int_0^1 \Phi(s) Fx(s)\mathrm{d}s \le \frac{d}{\mu} \int_0^1 \Phi(s)\mathrm{d}s < d.$$

This proves that $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$.

Now we need to show that condition (S_1) is satisfied. Take

$$x_0(t) = \frac{1}{2}\left(b + \frac{b}{\rho}\right), \;\; t \in J, \; \frac{b}{\rho} < d.$$

Then $x_0(t) > 0,\ t \in J$, and

$$\lambda[x_0] = \frac{1}{2}\left(b + \frac{b}{\rho}\right)\int_0^1 \mathrm{d}\Lambda(t) \geq 0.$$

Moreover,

$$\Theta(x_0) = \varphi(x_0) = \|x_0\| = \frac{b(\rho+1)}{2\rho} < \frac{b}{\rho} = c,\ \|x_0\| > b,$$
$$\Omega(x_0) = \min_{[\eta,1]} x_0(t) = \frac{b(\rho+1)}{2\rho} > b = \frac{b}{\rho}\,\rho > \rho\|x_0\|.$$

This proves that

$$\left\{x_0 \in P\left(\varphi, \Theta, \Omega, b, \frac{b}{\rho}, d\right):\ b < \Omega(x_0)\right\} \neq \emptyset.$$

Let $b \leq x(t) \leq \dfrac{b}{\rho}$ for $t \in [\eta, 1]$. Then also $b \leq x(\alpha(t)) \leq \dfrac{b}{\rho}$, because $t \leq \alpha(t) \leq 1$ for $t \in [\eta, 1]$. In view of Remark 2 and Assumption (A_2),

$$\Omega(Tx) = \min_{[\eta,1]}(Tx)(t) = \min_{[\eta,1]} \int_0^1 G_q(t,s)Fx(s)\mathrm{d}s$$
$$\geq \rho \int_0^1 \Phi(s)Fx(s)\mathrm{d}s \geq \frac{\rho b}{L}\int_0^1 \Phi(s)\mathrm{d}s > b.$$

This proves that condition (S_1) holds.

Now we need to prove that condition (S_2) is satisfied. Take $x \in P(\varphi, \Omega, b, d)$ and $\|Tx\| > \dfrac{b}{\rho} = c$. Then

$$\Omega(Tx) = \min_{[\eta,1]}(Tx)(t) \geq \rho\|Tx\| > \rho\,\frac{b}{\rho} = b,$$

so condition (S_2) holds.

Indeed, $\varphi(0) = 0 < a$, so $0 \notin R(\varphi, \Psi, a, d)$. Suppose that $x \in R(\varphi, \Psi, a, d)$ with $\Psi(x) = \|x\| = a$. By Remark 2 and condition (A_3), we get

$$\Psi(Tx) = \|Tx\| \leq \int_0^1 \Phi(s)Fx(s)\mathrm{d}s \leq \frac{a}{\mu}\int_0^1 \Phi(s)\mathrm{d}s < a.$$

This shows that condition (S_3) holds, which completes the proof.

Remark 4. If $f(t,0) \equiv 0$, then $x(t) \equiv 0$ is a solution of problem (1).

4. SOME COMMENTS

1. Remark 3 shows that the measure $d\Lambda$ can be a signed measure.

2. As the function α we can take, for example, $\alpha(t)=\sqrt{t}$ or $\alpha(t)=\sqrt[4]{t}$. Theorem 2 holds also in the case when $\alpha(t)=t,\ t\in J$.

3. Let

$$\lambda[x]=\sum_{i=1}^{m}\beta_i x(\gamma_i),\ \ 0<\gamma_1<\gamma_2<\cdots<\gamma_m<1,\ \ \beta_i\in\mathbb{R},\ i=1,2,\ldots,m.$$

In this case, we need the following conditions to be satisfied:

$$\sum_{i=1}^{m}\beta_i\geq 0,\quad 0\leq\sum_{i=1}^{m}\beta_i\gamma_i^{q-1}<\frac{\Gamma(q)}{\Gamma(q-k)},\quad \mathcal{G}(s)=\sum_{i=1}^{m}\beta_i G_1(\gamma_i,s)\geq 0,\ s\in J.$$

4. Let

$$\lambda[x]=\int_0^1 x(t)g(t)\mathrm{d}t,\quad g\in C(J,\mathbb{R}).$$

Now, we need the conditions:

$$\int_0^1 g(t)\mathrm{d}t\geq 0,\quad 0\leq\int_0^1 t^{q-1}g(t)\mathrm{d}t<\frac{\Gamma(q)}{\Gamma(q-k)},$$
$$\mathcal{G}(s)=\int_0^1 G_1(t,s)g(t)\mathrm{d}t\geq 0,\ s\in J.$$

5. An example, which also cowers multi–point and integral boundary conditions as a special case of functional λ is

$$\lambda[x]=\sum_{i=1}^{m}\beta_i x(\gamma_i)+\int_0^1 x(t)g(t)\mathrm{d}t,$$

where γ_i are distinct points in $(0,1)$ and $g\in C(J,\mathbb{R})$.

5. SPECIAL CASES OF PROBLEM (1)

For example, if $q=\frac{7}{2}$, then (1) reduces to the equation

$$D^{7/2}x(t)+f(t,x(\alpha(t)))=0,\quad t\in J_0 \tag{5}$$

with BCs

$$x^{(i)}(0)=0,\quad i=0,1,2,\quad x'(1)=\lambda[x] \tag{6}$$

with $k=1$, or

$$x^{(i)}(0)=0,\quad i=0,1,2,\quad D^{3/2}x(1)=\lambda[x] \tag{7}$$

if $k = \frac{3}{2}$. Then, Assumption H_2 takes the form

$$0 \le \int_0^1 t^{5/2} \mathrm{d}\Lambda(t) < \frac{5}{2}, \tag{8}$$

for BCs (6), and

$$0 \le \int_0^1 t^{5/2} \mathrm{d}\Lambda(t) < \frac{15}{8}\sqrt{\pi} \approx 3.3, \tag{9}$$

for BCs (7).

Moreover, the measure $\mathrm{d}\Lambda = (at - 1)\mathrm{d}t$ from Remark 3, both changes the sign and it satisfies Assumptions H_2, H_3 if

$$2 \le a < \frac{351}{28} \approx 12.5 \quad \text{in case of BCs (6)},$$
$$2 \le a < \frac{144 + 945\sqrt{\pi}}{112} \approx 16.2 \quad \text{in case of BCs (7)}.$$

Then, basing on Theorem 2, we can formulate the following results for problems (5) with BCs (6) or (7).

Theorem 3. *Put* $q = \frac{7}{2}$, $k = 1$. *Let all assumptions of Theorem* 2 *hold with* (8) *instead of Assumption* H_2. *Then the assertion of Theorem* 2 *holds for problem* (5)–(6).

Theorem 4. *Put* $q = \frac{7}{2}$, $k = \frac{3}{2}$. *Let all assumptions of Theorem* 2 *hold with* (9) *instead of Assumption* H_2. *Then the assertion of Theorem* 2 *holds for problem* (5)–(7).

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