Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **9** (2015), 271–284. doi:10.2298/AADM150811016M

EXISTENCE OF NONOSCILLATORY SOLUTIONS FOR SYSTEM OF NEUTRAL DIFFERENCE EQUATIONS

Małgorzata Migda, Ewa Schmeidel, Małgorzata Zdanowicz

The system of neutral type difference equations with delays

$$\begin{cases} \Delta\big(x(n) + p(n)\,x(n-\tau)\big) = a(n)\,f(y(n-\ell)) \\ \Delta y(n) = b(n)\,g(z(n-m)) \\ \Delta z(n) = c(n)\,h(x(n-k)) \end{cases}$$

is considered. The aim of this paper is to present sufficient conditions for the existence of nonoscillatory bounded positive solutions of the considered system with various $(p(n))$.

1. INTRODUCTION

In this paper we consider a nonlinear difference system of the three equations of the form

$$\begin{cases} \Delta\big(x(n) + p(n)\,x(n-\tau)\big) = a(n)\,f(y(n-\ell)) \\ \Delta y(n) = b(n)\,g(z(n-m)), \qquad\qquad n \in \mathbb{N}_0 \\ \Delta z(n) = c(n)\,h(x(n-k)) \end{cases} \tag{1}$$

where Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$, $(a(n))$, $(b(n))$, $(c(n))$ are sequences of real numbers, $(p(n))$ is a sequence of positive real numbers, τ, l,m, k are nonnegative integers, and functions $f, g, h \colon \mathbb{R} \to \mathbb{R}$. Here $\mathbb{R}$ is a set of real numbers and $\mathbb{N} = \{0, 1, 2, \dots\}$. By n_0 we denote $\max\{\tau, l, m, k\}$, and $\mathbb{N}_0 = \{n_0, n_0 + 1, \dots\}$. In the above system $(x(n))$, $(y(n))$ and $(z(n))$ are real sequences defined for $n \in \mathbb{N}$. Throughout this paper, $X(n)$ denotes vector

2010 Mathematics Subject Classification. 39A10, 39A22.
Keywords and Phrases. System of difference equations, three dimensional, neutral type, nonoscillatory solutions, boundedness.

$[x(n), y(n), z(n)]^T$. For the elements of $\mathbb{R}^3$ the symbol $|\cdot|$ stands for the maximum norm. By B we denote the Banach space of all bounded sequences in $\mathbb{R}^3$ with the supremum norm, i.e.,

$$B = \left\{X \colon \mathbb{N} \to \mathbb{R}^3 \colon \ \|X\| = \sup_{n\in\mathbb{N}} |X(n)| < \infty\right\}.$$

A sequence of real numbers is said to be nonoscillatory if it is either eventually positive or eventually negative. By a solution of system (1) we mean a sequence $(X(n))$ which satisfied system (1) for sufficiently large n. A solution X of the system (1) is called nonoscillatory if all its components, i.e. x, y, z are nonoscillatory.

Note, that system (1) includes different types of third order difference equations. For example, if the sequences a, b are positive, f, g are linear functions and $l = 0$, $m = 0$, the system (1) reduces to the third-order neutral type difference equation of the form

$$\Delta\left(\frac{1}{b(n)}\Big(\Delta\frac{1}{a(n)}\Delta(x(n) + p(n)x(n-\tau))\Big)\right) = c(n)h(x(n-k)).$$

Such equations and their special cases have been studied by many authors, see for example, [**2**], [**3**], [**4**], [**7**], [**9**], [**12**], [**18**] and the references cited therein.

The existence of a bounded nonoscillatory solution of nonlinear neutral type difference system of two second-order equations has been studied in [**17**].

Some oscillation results, classification of nonoscillatory solutions or boundedness criteria for system (1) have been presented in [**13**], [**14**], [**15**] and [**16**] under the assumption

$$\sum_{n=1}^{\infty} a(n) = \sum_{n=1}^{\infty} b(n) = \infty.$$

In this paper we consider the case when

$$\sum_{n=1}^{\infty} |a(n)| < \infty, \quad \sum_{n=1}^{\infty} |b(n)| < \infty, \quad \sum_{n=1}^{\infty} |c(n)| < \infty.$$

We establish sufficient conditions for the existence of nonoscillatory bounded positive solutions of the considered system with various $(p(n))$. The results are illustrated by examples.

The following definition and theorems will be used in the sequel.

Definition 1 (Uniformly Cauchy subset, [**6**]). *A subset Ω of the Banach space B is said to be uniformly Cauchy if for every $\varepsilon > 0$ there exists a positive integer N such that $|X(i) - X(j)| < \varepsilon$ whenever $i, j > N$ for any $X \in \Omega$.*

Theorem 1 (Arzelá–Ascoli's Theorem, [**1**]). *A bounded and uniformly Cauchy subset of B is relatively compact.*

Theorem 2 (Krasnoselskii's Fixed Point Theorem, [**8**]). *Let B be a Banach space, let Ω be a bounded closed convex subset of B and let F, T be maps of Ω into B such that $Fx + Ty \in \Omega$ for every pair $x, y \in \Omega$. If F is a contraction and T is completely continuous, then the equation $Fx + Tx = x$ has a solution in Ω.*

Theorem 3 (Schauder's Fixed Point Theorem, [**5**]). *Let M be a nonempty, compact and convex subset of a Banach space and let $T\colon M \to M$ be a continuous. Then T has a fixed point in M.*

2. MAIN RESULTS

In this section, using the Krasnoselskii's fixed point theorem and Schauder fixed point theorem, we establish sufficient conditions for the existence of nonoscillatory bounded solutions of system (1).

Theorem 4. *Assume that*

$$\sum_{n=1}^{\infty}|a(n)|<\infty,\qquad \sum_{n=1}^{\infty}|b(n)|<\infty,\qquad \sum_{n=1}^{\infty}|c(n)|<\infty, \tag{2}$$

$$f,g,h\colon \mathbb{R}\to\mathbb{R} \text{ are continuous functions.} \tag{3}$$

If there exists a real number c_p such that

$$0<p(n)\le c_p<1,\qquad n\in\mathbb{N}_0, \tag{4}$$

then system (1) *has a bounded nonoscillatory solution.*

Proof. For a fixed positive real number r we define the set

$$\Omega_1=\{X\in B\colon\ x(n),y(n),z(n)\in I_1,\ n\in\mathbb{N}\},$$

where $I_1=\left[\frac{1}{3}(1-c_p)r,r\right]$. Ω_1 is bounded closed convex subset of the Banach space B. Since condition (3) is satisfied, we can set

$$\begin{aligned} M_f &= \max\{|f(t)|\colon\ t\in I_1\},\\ M_g &= \max\{|g(t)|\colon\ t\in I_1\},\\ M_h &= \max\{|h(t)|\colon\ t\in I_1\}. \end{aligned}$$

From (2), there exists $n_1\in\mathbb{N}_0$ such that

$$\sum_{n=n_1}^{\infty}|a(n)|\le\frac{(1-c_p)r}{3M_f},\quad \sum_{n=n_1}^{\infty}|b(n)|\le\frac{(1-c_p)r}{3M_g},\quad \sum_{n=n_1}^{\infty}|c(n)|\le\frac{(1-c_p)r}{3M_h}.$$

Next, we define the maps $F,T\colon\Omega_1\to B$ where

$$F=\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix},\qquad T=\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix},$$

as follows

$$(FX)(n)=\begin{bmatrix}-p(n)x(n-\tau)+\dfrac{(2+c_p)r}{3}\\ \dfrac{2(1-c_p)r}{3}\\ \dfrac{2(1-c_p)r}{3}\end{bmatrix}\quad\text{for}\quad n\ge n_1,$$

and

$$(FX)(n) = (FX)(n_1) \text{ for } 0 \le n < n_1;$$

(5) $$(TX)(n) = \begin{bmatrix} -\sum_{s=n}^{\infty} a(s)\, f(y(s-\ell)) \\ -\sum_{s=n}^{\infty} b(s)\, g(z(s-m)) \\ -\sum_{s=n}^{\infty} c(s)\, h(x(s-k)) \end{bmatrix} \quad \text{for} \quad n \ge n_1,$$

and

$$(TX)(n) = (TX)(n_1) \text{ for } 0 \le n < n_1.$$

We will show that F and T satisfy the conditions of Theorem 2. First we show that if $X, \bar{X} \in \Omega_1$, then $FX + T\bar{X} \in \Omega_1$. For $n \ge n_1$ we have

$$\begin{aligned} (F_1X)(n) + (T_1\bar{X})(n) &= -p(n)\, x(n-\tau) + \frac{(2+c_p)r}{3} - \sum_{s=n}^{\infty} a(s)\, f(\bar{y}(s-\ell)) \\ &\le \frac{(2+c_p)r}{3} + \sum_{s=n}^{\infty} \left|a(s)\right| \left|f(\bar{y}(s-\ell))\right| \\ &\le \frac{2}{3}r + \frac{1}{3}c_p r + M_f \cdot \frac{(1-c_p)r}{3M_f} = r, \\ (F_1X)(n) + (T_1\bar{X})(n) &= -p(n)\, x(n-\tau) + \frac{(2+c_p)r}{3} - \sum_{s=n}^{\infty} a(s)\, f(\bar{y}(s-\ell)) \\ &\ge \frac{(2+c_p)r}{3} - \sum_{s=n}^{\infty} \left|a(s)\right| \left|f(\bar{y}(s-\ell))\right| - p(n)\, x(n-\tau) \\ &\ge \frac{2}{3}r + \frac{1}{3}c_p r - M_f \cdot \frac{(1-c_p)r}{3M_f} - c_p r \\ &= \frac{2}{3}r + \frac{1}{3}c_p r - \frac{1}{3}r + \frac{1}{3}c_p r - c_p r = \frac{1}{3}(1-c_p)r. \end{aligned}$$

Below we present reasoning for F_2; the same conclusions can be drawn for F_3. For $n \ge n_1$

$$\begin{aligned} (F_2X)(n) + (T_2\bar{X})(n) &= \frac{2(1-c_p)r}{3} - \sum_{s=n}^{\infty} b(s)\, g(\bar{z}(s-m)) \\ &\le \frac{2(1-c_p)r}{3} + \sum_{s=n}^{\infty} \left|b(s)\right| \left|g(\bar{z}(s-m))\right| \\ &\le \frac{2}{3}r - \frac{2}{3}c_p r + M_g \cdot \frac{(1-c_p)r}{3M_g} = (1-c_p)r \le r, \\ (F_2X)(n) + (T_2\bar{X})(n) &= \frac{2(1-c_p)r}{3} - \sum_{s=n}^{\infty} b(s)\, g(\bar{z}(s-m)) \end{aligned}$$

$$\geq \frac{2(1-c_p)r}{3} - \sum_{s=n}^{\infty} \left|b(s)\right| \left|g(\bar{z}(s-m))\right|$$
$$\geq \frac{2}{3}r - \frac{2}{3}c_p r - M_g \cdot \frac{(1-c_p)r}{3M_g} = \frac{1}{3}(1-c_p)r.$$

The task is now to prove that F is a contraction mapping. It is easy to see

$$\begin{aligned} \left|(F_1X)(n) - (F_1\bar{X})(n)\right| &\leq p(n)\,|x(n-\tau) - \bar{x}(n-\tau)| \\ &\leq c_p\,|x(n-\tau) - \bar{x}(n-\tau)|, \\ \left|(F_2X)(n) - (F_2\bar{X})(n)\right| &= 0, \\ \left|(F_3X)(n) - (F_3\bar{X})(n)\right| &= 0, \end{aligned}$$

for $X, \bar{X} \in \Omega_1$ and $n \geq n_1$. Hence

$$\|FX - F\bar{X}\| \leq c_p \|X - \bar{X}\|,$$

where, by (4), there is $0 < c_p < 1$.

The next step is to show continuity of T. Let $X_j = [x_j, y_j, z_j]^T \in \Omega_1$ for any $j \in \mathbb{N}$, and let $(X_j(n))$ be such that $x_j(n) \to x(n)$, $y_j(n) \to y(n)$, $z_j(n) \to z(n)$ as $j \to \infty$. Since $X_j \in \Omega_1$ and Ω_1 is closed, we have $X = [x, y, z]^T \in \Omega_1$, so $x(n), y(n), z(n) \in I_1$ for $n \in \mathbb{N}$. By (5), (2) and (3) we obtain

$$|(T_1X_j)(n) - (T_1X)(n)| \leq \sum_{s=n}^{\infty} \left|a(s)\right| \left|f(y_j(s-\ell)) - f(y(s-\ell))\right| \to 0 \text{ if } j \to \infty,$$

for any $n \in \mathbb{N}$. Analogously

$$|(T_2X_j)(n) - (T_2X)(n)| \to 0 \text{ and } |(T_3X_j)(n) - (T_3X)(n)| \to 0 \text{ if } j \to \infty.$$

Therefore

$$\|(TX_j) - (TX)\| \to 0 \text{ if } j \to \infty.$$

We get that T is a continuous mapping.

Next we will demonstrate that $T\Omega_1$ is uniformly Cauchy. As before we only show transformations for T_1, since similar arguments apply to T_2 and T_3. Let $X \in \Omega_1$. We conclude from the assumptions (2) and (3) that for any given $\varepsilon > 0$ there exists an integer $n_2 > n_1$ such that for $n \geq n_2$

$$\sum_{s=n}^{\infty} \left|a(s)\right| \left|f(y(s-\ell))\right| < \frac{\varepsilon}{2}.$$

Hence for $n_4 > n_3 \geq n_2$ we obtain

$$|(T_1X)(n_4) - (T_1X)(n_3)| = \left| \sum_{s=n_4}^{\infty} a(s)\ f(y(s-\ell)) - \sum_{s=n_3}^{\infty} a(s)\ f(y(s-\ell)) \right| < \varepsilon.$$

Therefore $T\Omega_1$ is uniformly Cauchy.

By Theorem 2, there exists $(X(n))$ such that $(FX)(n)+(TX)(n) = X(n)$. Finally, we verify that $X(n)$ satisfies system (1) for $n \geq n_1$. As $(F_1X)(n)+(T_1X)(n) = x(n)$ we have

$$-p(n)\,x(n-\tau) + \frac{(2+c_p)r}{3} - \sum_{s=n}^{\infty} a(s)f(y(s-\ell)) = x(n).$$

Next we move the term $-p(n)\,x(n-\tau)$ to the right hand side of the equation. Using the forward difference operator to the obtained equation we get

$$\Delta\left(x(n) + p(n)\,x(n-\tau)\right) = -\Delta \sum_{s=n}^{\infty} a(s)f(y(s-\ell)).$$

Hence

$$\Delta\left(x(n) + p(n)\,x(n-\tau)\right) = a(n)f(y(n-\ell)).$$

Similarly, if $(F_2X)(n) + (T_2X)(n) = y(n)$, then

$$\frac{2(1-c_p)r}{3} - \sum_{s=n}^{\infty} b(s)\; g(z(s-m)) = y(n).$$

Using the forward difference operator we get

$$\Delta y(n) = -\sum_{s=n+1}^{\infty} b(s)\; g(z(s-m)) + \sum_{s=n}^{\infty} b(s)\; g(z(s-m)),$$

and hence

$$\Delta y(n) = b(n)\; g(z(n-m)).$$

In the same manner we verify that $(F_3X)(n) + (T_3X)(n) = z(n)$ implies the third equation of (1). The proof is complete.

Example 1. Consider the difference system

$$\begin{aligned}
&\Delta\left(x(n) + \frac{1}{2n}\,x(n-1)\right) = \frac{-3n^2+n}{4n^4-2n^2-2}\,y(n)\\
&\Delta y(n) = \frac{-2n^2+n+1}{2n^5+n^4-4n^3-3n^2}\,z(n-1)\\
&\Delta z(n) = \frac{n-1}{n^3+n^2}\,x(n-1).
\end{aligned}$$

All assumptions of Theorem 4 are satisfied. One of the bounded solutions of the above system is $X(n) = \left[1+\frac{1}{n}, 2+\frac{1}{n^2}, 2-\frac{1}{n}\right]^T$.

Theorem 5. *Assume that conditions* (2) *and* (3) *are satisfied. If there exists a real number* $\tilde{c_p}$ *such that*

$$1 < \tilde{c_p} \leq p(n), \quad n \in \mathbb{N}_0, \tag{6}$$

then system (1) *has a bounded nonoscillatory solution.*

Proof. We define subset Ω_2 of B in the following way

$$\Omega_2 = \{X \in B\colon\ x(n), y(n), z(n) \in I_2,\ n \in \mathbb{N}_0\},$$

where $I_2 = \left[\frac{1}{3}(\tilde{c_p} - 1)r, \tilde{c_p} r\right]$, r is a fixed positive real number. Obviously Ω_2 is a bounded, closed and convex subset of B. Let us set

$$\begin{aligned}\tilde{M}_f &= \max\{|f(t)|\colon\ t \in I_2\},\\ \tilde{M}_g &= \max\{|g(t)|\colon\ t \in I_2\},\\ \tilde{M}_h &= \max\{|h(t)|\colon\ t \in I_2\}.\end{aligned}$$

From the assumption (2), we conclude that there exists $n_1 \in \mathbb{N}_0$ such that

$$\sum_{n=n_1}^{\infty} |a(n)| \le \frac{(\tilde{c_p} - 1)r}{3\tilde{M}_f},\ \sum_{n=n_1}^{\infty} |b(n)| \le \frac{(\tilde{c_p} - 1)r}{3\tilde{M}_g},\ \sum_{n=n_1}^{\infty} |c(n)| \le \frac{(\tilde{c_p} - 1)r}{3\tilde{M}_h}.$$

We define the maps $F, T\colon \Omega_2 \to B$ in the following way

$$(FX)(n) = \begin{bmatrix} -\dfrac{x(n+\tau)}{p(n+\tau)} + \dfrac{(2\tilde{c_p}+1)r}{3} \\ \dfrac{(2\tilde{c_p}-1)r}{3} \\ \dfrac{(2\tilde{c_p}-1)r}{3} \end{bmatrix} \quad \text{for} \quad n \ge n_1$$

and

$$(FX)(n) = (FX)(n_1) \ \text{ for } \ 0 \le n < n_1;$$

$$(TX)(n) = \begin{bmatrix} -\dfrac{1}{p(n+\tau)} \displaystyle\sum_{s=n+\tau}^{\infty} a(s) f(y(s-\ell)) \\ -\displaystyle\sum_{s=n}^{\infty} b(s)\, g(z(s-m)) \\ -\displaystyle\sum_{s=n}^{\infty} c(s)\, h(x(s-k)) \end{bmatrix} \quad \text{for} \quad n \ge n_1$$

and

$$(TX)(n) = (TX)(n_1) \ \text{ for } \ 0 \le n < n_1.$$

Let $X, \bar{X} \in \Omega_2$, $n \ge n_1$. To prove $FX + T\bar{X} \in \Omega_2$, we will present all transformations only for the first and the second components of F and T. For $n \ge n_1$ we have

$$\begin{aligned}(F_1 X)(n) + (T_1 \bar{X})(n) = &-\frac{1}{p(n+\tau)}\, x(n+\tau) + \frac{(2\tilde{c_p}+1)r}{3}\\ &- \frac{1}{p(n+\tau)} \sum_{s=n+\tau}^{\infty} a(s) f(\bar{y}(s-\ell))\end{aligned}$$

$$\begin{aligned}&\le \frac{(2\tilde{c_p}+1)r}{3} + \frac{1}{p(n+\tau)} \sum_{s=n+\tau}^{\infty} |a(s)|\,|f(\bar{y}(s-l))| \\ &\le \frac{2}{3}\tilde{c_p}r + \frac{1}{3}r + \tilde{M}_f \cdot \frac{(\tilde{c_p}-1)r}{3\tilde{M}_f} = \tilde{c_p}r,\end{aligned}$$

and

$$\begin{aligned}(F_1X)(n) + (T_1\bar{X})(n) &= -\frac{1}{p(n+\tau)}\, x(n+\tau) + \frac{(2\tilde{c_p}+1)r}{3} \\ &\quad - \frac{1}{p(n+\tau)} \sum_{s=n+\tau}^{\infty} a(s) f(\bar{y}(s-\ell)) \\ &\ge \frac{(2\tilde{c_p}+1)r}{3} - \sum_{s=n}^{\infty} \big|a(s)\big|\,\big|f(\bar{y}(s-\ell))\big| - \frac{1}{p(n+\tau)}\, x(n+\tau) \\ &\ge \frac{2}{3}\tilde{c_p}r + \frac{1}{3}r - \tilde{M}_f \cdot \frac{(\tilde{c_p}-1)r}{3\tilde{M}_f} - r = \frac{1}{3}(\tilde{c_p}-1)r.\end{aligned}$$

Similarly,

$$\begin{aligned}(F_2X)(n) + (T_2\bar{X})(n) &= \frac{(2\tilde{c_p}-1)r}{3} - \sum_{s=n}^{\infty} b(s) g(\bar{z}(s-m)) \\ &\le \frac{(2\tilde{c_p}-1)r}{3} + \sum_{s=n}^{\infty} \big|b(s)\big|\,\big|g(\bar{z}(s-m))\big| \\ &\le \frac{2}{3}\tilde{c_p}r - \frac{1}{3}r + \tilde{M}_g \cdot \frac{(\tilde{c_p}-1)r}{3\tilde{M}_g} = \left(\tilde{c_p} - \frac{2}{3}\right) r \le \tilde{c_p}r,\end{aligned}$$

and

$$\begin{aligned}(F_2X)(n) + (T_2\bar{X})(n) &= \frac{(2\tilde{c_p}-1)r}{3} - \sum_{s=n}^{\infty} b(s) g(\bar{z}(s-m)) \\ &\ge \frac{(2\tilde{c_p}-1)r}{3} - \sum_{s=n}^{\infty} \big|b(s)\big|\,\big|g(\bar{z}(s-m))\big| \\ &\ge \frac{2}{3}\tilde{c_p}r - \frac{1}{3}r - \tilde{M}_g \cdot \frac{(\tilde{c_p}-1)r}{3\tilde{M}_g} = \frac{1}{3}\tilde{c_p}r \ge \frac{1}{3}(\tilde{c_p}-1)r.\end{aligned}$$

To see that F is a contraction mapping, let us observe that

$$\begin{aligned}\big|(F_1X)(n) - (F_1\bar{X})(n)\big| &\le \frac{1}{p(n+\tau)} \big|x(n+\tau) - \bar{x}(n+\tau)\big| \\ &\le \frac{1}{\tilde{c_p}} \big|x(n+\tau) - \bar{x}(n+\tau)\big|\end{aligned}$$

and

$$\big|(F_2X)(n) - (F_2\bar{X})(n)\big| = 0, \qquad \big|(F_3X)(n) - (F_3\bar{X})(n)\big| = 0.$$

Hence

$$\|FX - F\bar{X}\| \le \frac{1}{\tilde{c_p}} \|X - \bar{X}\|,$$

but $\frac{1}{\tilde{c_p}} < 1$ by (6).

The proof of the continuity of the mapping T goes exactly in the same way as previously.

By virtue of Theorem 2, there exists $(X(n))$ such that $(FX)(n) + (TX)(n) = X(n)$. Finally, we show that such $(X(n))$ satisfy the system (1) for $n \ge n_1$. Let $(F_1X)(n) + (T_1X)(n) = x(n)$. Thus

$$-\frac{x(n+\tau)}{p(n+\tau)} + \frac{(2\tilde{c_p}+1)r}{3} - \frac{1}{p(n+\tau)} \sum_{s=n+\tau}^{\infty} a(s) f(y(s-\ell)) = x(n).$$

Therefore

$$\Delta\left(x(n) + \frac{x(n+\tau)}{p(n+\tau)}\right) = -\Delta\left(\frac{1}{p(n+\tau)} \sum_{s=n+\tau}^{\infty} a(s) f(y(s-\ell))\right).$$

Then, by obtaining a common denominator, we have

$$\begin{aligned} (7) \qquad & \frac{1}{p(n+\tau+1)} \Delta\big(x(n+\tau) + p(n+\tau)x(n)\big) \\ & \quad + \left(\Delta \frac{1}{p(n+\tau)}\right) \big(x(n+\tau) + p(n+\tau)x(n)\big) \\ & = -\frac{1}{p(n+\tau+1)} \Delta\left(\sum_{s=n+\tau}^{\infty} a(s) f(y(s-\ell))\right) \\ & \quad - \left(\Delta \frac{1}{p(n+\tau)}\right)\left(\sum_{s=n+\tau}^{\infty} a(s) f(y(s-\ell))\right). \end{aligned}$$

Since

$$-\Delta\left(\sum_{s=n+\tau}^{\infty} a(s) f(y(s-\ell))\right) = a(n+\tau) f(y(n+\tau-\ell)),$$

from (7) we get

$$\Delta\big(x(n+\tau) + p(n+\tau)x(n)\big) = a(n+\tau) f(y(n+\tau-\ell)).$$

Now we can transform the last equation into

$$\Delta\big(x(n) + p(n)x(n-\tau)\big) = a(n) f(y(n-\ell)).$$

Assume that $(F_2X)(n) + (T_2X)(n) = y(n)$. It is easier than above to see that X satisfies (1). In fact, then we have

$$\frac{(2\tilde{c_p}-1)r}{3} - \sum_{s=n}^{\infty} b(s)\, g(z(s-m)) = y(n).$$

Then acting with the forward difference operator and simplifying, we arrive to

$$\Delta y(n) = -\sum_{s=n+1}^{\infty} b(s)\; g(z(s-m)) + \sum_{s=n}^{\infty} b(s)\; g(z(s-m)),$$

and hence

$$\Delta y(n) = b(n)\; g(z(n-m)).$$

Using the third equation $(F_3X)(n) + (T_3X)(n) = z(n)$ we conclude in exactly the same manner. The proof is now complete. □

Note, that the assumptions of Theorem 2 and Theorem 3 ensure, that system (1) has not only one bounded nonoscillatory solution, but uncountably many bounded nonoscillatory solutions.

Example 2. Now, let us consider the difference system

$$\begin{aligned}
&\Delta\left(x(n) + \left(2 + \frac{1}{2^n}\right)x(n-2)\right) = \frac{1}{2}\cdot\frac{7\cdot 2^n + 6}{4^n + 2^{n+1}}\, y(n-1)\\
&\Delta y(n) = -\frac{1}{16}\frac{4^n}{(3\cdot 2^{n-1}+1)^3}\, z^3(n-1)\\
&\Delta z(n) = -\frac{1}{32}\frac{2^n}{(2^{n-1}-1)^2}\, x^2(n-2).
\end{aligned}$$

It is easy to see that all assumptions of Theorem 5 are satisfied. Hence the above system has bounded solutions. One of such solutions is

$$X(n) = \left[2 - \frac{1}{2^n}, 1 + \frac{1}{2^n}, 3 + \frac{1}{2^n}\right]^T.$$

In the next theorem, for the special case $p(n) \equiv 1$, we prove an even better result. We give sufficient conditions under which for any real constants d_1, d_2, d_3 there exists a solution of system (1) convergent to $[d_1, d_2, d_3]^T$.

Theorem 6. *Assume that conditions* (2) *and* (3) *are satisfied. If* $p(n) \equiv 1$, *then for any real constants* d_1, d_2, d_3 *there exists a solution* $(X(n))$ *of the system* (1) *such that* $\lim_{n\to\infty} X(n) = [d_1, d_2, d_3]^T$.

Proof. Let $d_1, d_2, d_3 \in \mathbb{R}$ and let us choose a real number e such that $e > 0$. There exist constants $M_i > 1$, $i = 1, 2, 3$ such that

$$\begin{aligned}
|f(t)| &\le M_1 \quad \text{for every} \quad t \in [d_1 - e, d_1 + e],\\
|g(t)| &\le M_2 \quad \text{for every} \quad t \in [d_2 - e, d_2 + e],\\
|h(t)| &\le M_3 \quad \text{for every} \quad t \in [d_3 - e, d_3 + e].
\end{aligned}$$

Let us denote $M = \max\{M_1, M_2, M_3\}$ and

$$S_a(n) = \sum_{j=n}^{\infty} |a(j)|,\;\; S_b(n) = \sum_{j=n}^{\infty} |b(j)|,\;\; S_c(n) = \sum_{j=n}^{\infty} |c(n)|.$$

By (2), there exists an index $n_1 \geq n_0$ such that for $n \geq n_1$ we have

$$S_a(n) \leq \frac{e}{M}, \quad S_b(n) \leq \frac{e}{M}, \quad S_c(n) \leq \frac{e}{M}.$$

We define a subset Ω_3 of B by

$$\Omega_3 = \{X \in B \colon X(0) = \cdots = X(n_1 - 1) = D \text{ and } |X(n) - D| \leq M\, |S(n)| \text{ for } n \geq n_1\},$$

where $D = [d_1, d_2, d_3]^T$ and $S = [S_a, S_b, S_c]^T$. It is easy to check, that Ω_3 is a convex subset of B. It can be also shown that Ω_3 is compact (see, for example, the proofs of Theorem 1 in [**10**] or Lemma 4.7 in [**11**]). Now, for $n \geq 0$, we define a map

$$T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \colon \Omega_3 \to B$$

as follows

$$(T_1X)(n) = \begin{cases} d_1 & \text{for } n < n_1 \\ d_1 - \displaystyle\sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau}^{n+2j\tau-1} a(s) f\left(y(s-\ell)\right) & \text{for } n \geq n_1 \quad \text{and } \tau > 0, \\ d_1 - \dfrac{1}{2} \displaystyle\sum_{s=n}^{\infty} a(s) f\left(y(s-\ell)\right) & \text{for } n \geq n_1 \quad \text{and } \tau = 0, \end{cases}$$

$$(T_2X)(n) = \begin{cases} d_2 & \text{for } n < n_1 \\ d_2 - \displaystyle\sum_{s=n}^{\infty} b(s)\, g(z(s-m)) & \text{for } n \geq n_1 \end{cases}$$

and

$$(T_3X)(n) = \begin{cases} d_3 & \text{for } n < n_1 \\ d_3 - \displaystyle\sum_{s=n}^{\infty} c(s)\, h(x(s-k)) & \text{for } n \geq n_1. \end{cases}$$

We will show that $T(\Omega_3) \subseteq \Omega_3$. It is easy to check, that

$$\sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau}^{n+2j\tau-1} |a(s)| \leq \sum_{s=n}^{\infty} |a(s)|\,. \tag{8}$$

Moreover, if $X \in \Omega_3$, then by the definition of Ω_3 we get $|y(n) - d_2| \leq e$ for all $n \in \mathbb{N}$. Hence $|f(y(n))| \leq M_1$ for every $(X(n)) \in \Omega_3$, $n \in \mathbb{N}$. Therefore by (8), for $n \geq n_1$ and $\tau > 0$, we get

$$|(T_1X)(n) - d_1| = \left| \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau}^{n+2j\tau-1} a(s) f\left(y(s-\ell)\right) \right| \tag{9}$$

$$\leq M_1 \sum_{s=n}^{\infty} |a(s)| \leq M S_a(n).$$

For $n \geq n_1$ and $\tau = 0$, we have

$$|(T_1 X)(n) - d_1| = \left| \frac{1}{2} \sum_{s=n}^{\infty} a(s) f\left(y(s-\ell)\right) \right| \leq M S_a(n).$$

Similarly, we get

$$|(T_2 X)(n) - d_2| \leq M S_b(n) \quad \text{and} \quad |(T_3 X)(n) - d_3| \leq M S_c(n). \tag{10}$$

So, $T(X) \in \Omega_3$ for every $X \in \Omega_3$ and $T(\Omega_3) \subseteq \Omega_3$. Similarly as in the proof of Theorem 4, it can be shown that T is continuous. By Schauder's fixed point theorem there exists $X \in \Omega_3$ such that $T(X) = X$, which is a solution of system (1). In fact, for $n \geq n_1$ and $\tau > 0$, we have

$$x(n) = d_1 - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau}^{n+2j\tau-1} a(s) f\left(y(s-\ell)\right),$$

$$x(n-\tau) = d_1 - \sum_{j=1}^{\infty} \sum_{s=n+(2j-2)\tau}^{n+(2j-1)\tau-1} a(s) f\left(y(s-\ell)\right).$$

Hence

$$x(n)+x(n-\tau) = 2d_1 - \sum_{j=1}^{\infty} \sum_{s=n+2(j-1)\tau}^{n+2j\tau-1} a(s) f\left(y(s-\ell)\right) = 2d_1 - \sum_{s=n}^{\infty} a(s) f\left(y(s-\ell)\right).$$

Therefore

$$\Delta\left(x(n) + x(n-\tau)\right) = - \sum_{s=n+1}^{\infty} a(s) f\left(y(s-\ell)\right) + \sum_{s=n}^{\infty} a(s) f\left(y(s-\ell)\right),$$

and hence

$$\Delta\left(x(n) + x(n-\tau)\right) = a(n) f\left(y(n-\ell)\right).$$

In the case $\tau = 0$, we obtain

$$\Delta\left(x(n) + x(n)\right) = 2\Delta x(n) = 2\Delta\left(d_1 - \frac{1}{2} \sum_{s=n}^{\infty} a(s) f\left(y(s-\ell)\right)\right) = a(n) f\left(y(n-\ell)\right).$$

Similarly, we get

$$\Delta y(n) = b(n)\, g(z(n-m)), \quad \Delta z(n) = c(n)\, h(x(n-k)).$$

This means that the sequence $(X(n))$ fulfills system (1) for $n \geq n_1$. By (2), the sequences S_a, S_b, and S_c tend to zero. Hence and from (9), (10) we get $\lim_{n\to\infty} X(n) = [d_1, d_2, d_3]^T$. This completes the proof.

Example 3. Let us consider the following system

$$\Delta\left(x(n)+x(n-2)\right) = -\frac{20}{9}\cdot\frac{1}{2\cdot 3^{n-1}+1}\,y(n-1)$$
$$\Delta y(n) = -\frac{2}{27}\cdot\frac{3^n}{(3^n-1)^2}\,z^2(n-1)$$
$$\Delta z(n) = \frac{2}{3}\cdot\frac{1}{4\cdot 3^n+9}\,x(n-2).$$

All assumptions of Theorem 6 are satisfied. It is easy to check that

$$X(n) = \left[4+\frac{1}{3^n}, 2+\frac{1}{3^n}, 3-\frac{1}{3^n}\right]^T$$

is the solution of the above system having the property $\lim_{n\to\infty} X(n) = [4,2,3]^T$.

We remark that the results obtained for system (1) can be extended analogically for a system of the form

$$\begin{cases} \Delta\big(x(n)+p_1(n)\,x(n-\tau_1)\big) = a(n)\,f(y(n-\ell)) \\ \Delta\big(y(n)+p_2(n)\,y(n-\tau_2)\big) = b(n)\,g(z(n-m)), n\in\mathbb{N}_0. \\ \Delta\big(z(n)+p_3(n)\,z(n-\tau_3)\big) = c(n)\,h(x(n-k)) \end{cases}$$

Acknowledgments. The first author was supported by the project PB-43-081/14DS of Ministry of Science and Higher Education of Poland.

REFERENCES

1. R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan: *Discrete Oscillation Theory.* Hindawi Publishing Corporation, New York, 2005.
2. A. Andruch-Sobiło, M. Migda: *On the oscillation of solutions of third order linear difference equations of neutral type.* Math. Bohem., **130** (2005), 19–33.
3. A. Andruch-Sobiło, M. Migda: *Bounded solutions of third order nonlinear difference equations,* Rocky Mountain J. Math., **36** (1), (2006), 23–34.
4. M. F. Aktas, A. Tiryaki, A. Zafer: *Oscillation of the third order nonlinear delay difference equations.* Turkish J. Math., **36** (2012), 422–436.
5. A. Burton: *Stability by Fixed Point Theory for Functional Differential Equations.* Dover Publications, 2006.
6. S. S. Cheng, W. T. Patula: *An existence theorem for a nonlinear difference equation.* Nonlinear Anal., **20** (1993), 193–203.
7. Z. Došlá, A. Kobza: *On third-order linear difference equations involving quasi-differences.* Adv. Difference Equ., 2006, Art. 65652, 1–13.
8. L.H. Erbe, Q. Kong, B. G. Zhang: *Oscillation theory for functional differential equations.* CRC Press, 1995
9. Z. Liu, M. Jia, S. M. Kang, Y. C. Kwun: *Bounded positive solutions for a third order discrete equation.* Abstr. Appl. Anal., **2012** (2012), Article ID 237036, 12 pages.

10. J. Migda: *Asymptotic behavior of solutions of nonlinear difference equations.* Math. Bohem., **129** (4), (2004), 349–359.
11. J. Migda: *Approximative solutions of difference equations.* Electron. J. Qual. Theory Differ. Equ., **13** (2014), 1–26.
12. J. Migda: *Approximative solutions to difference equations of neutral type.* Appl. Math. Comput., **268** (2015), 763-774.
13. E. Schmeidel: *Boundedness of solutions of nonlinear three-dimensional difference systems with delays.* Fasc. Math., **44** (2010), 107–113.
14. E. Schmeidel: *Oscillation of nonlinear third-dimensional difference systems with delays.* Math. Bohem., **135** (2010), 163–170.
15. E. Schmeidel: *Properties of solutions of system of difference equations with neutral term.* Funct. Differ. Equ., **18** (3–4), (2011), 287–296.
16. E. Thandapani, B. Ponnammal: *Oscillatory properties of solutions of three dimensional difference systems.* Math. Comput. Modelling, **42** (5–6), 2005, 641–650.
17. E. Thandapani, R. Karunakaran, I. M. Arockiasamy: *Bounded nonoscillatory solutions of neutral type difference systems.* Electron. J. Qual. Theory Differ Equ., Spec. Ed. I, **25** (2009), 1–8.
18. E. Thandapani, M. Vijaya, T. Li: *On the oscillation of third order half-linear neutral type difference equations.* Electron. J. Qual. Theory Differ. Equ. **76** (2011), 1–13.

Poznan University of Technology
Poland
E-mail: malgorzata.migda@put.poznan.pl

(Received January 6, 2015)
(Revised August 11, 2015)

University of Bialystok
Poland
E-mails: eschmeidel@math.uwb.edu.pl
mzdan@math.uwb.edu.pl