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GRAPHS WITH NO INDUCED WHEEL AND NO INDUCED ANTIWHEEL

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A wheel is a graph that consists of a chordless cycle of length at least 4 plus a vertex with at least three neighbors on the cycle. An antiwheel is the complementary graph of a wheel. It was shown recently that detecting induced wheels is an NP-complete problem. In contrast, it is shown here that graphs that contain no wheel and no antiwheel have a very simple structure and consequently can be recognized in polynomial time.

1. INTRODUCTION

Four families of graphs have repeatedly played important roles in structural graph theory recently. They are called *Truemper configurations* as they were first used by Truemper in several theorems [**9**]. These configurations are called *pyramids*, *prisms*, *thetas* and *wheels*. We will not recall all the definitions, as we do not need all of them here; see Vušković [**10**] for an extensive survey on Truemper configurations and their important role in graph theory. It is interesting to know the complexity of deciding whether a graph contains (as an induced subgraph) a Truemper configuration of a certain type. The problem is polynomial for pyramids [**1**]; indeed it is one of the main steps in the polynomial-time recognition algorithm for perfect graphs [**1**]. It is also polynomial for thetas [**2**]. On the other hand, the problem is NP-complete prisms [**7**].

Here we will deal with the fourth Truemper configuration, the wheel. A *wheel* is a graph that consists of a chordless cycle of length at least 4 plus a vertex that has at least three neighbors on the cycle. An *antiwheel* is the complementary graph of a wheel. Diot, Tavenas and Trotignon [**3**] proved that it is also NP-complete

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to decide if a graph contains a wheel as an induced subgraph. They mention the open question of characterizing the graphs that contain no wheel and no antiwheel. We solve this question here by giving a complete description of the structure of these graphs, from which it follows that they can be recognized in linear time.

We use the standard graph-theoretic terminology. We let K_n, P_n and C_n respectively denote the complete graph, path and cycle on n vertices. The *length* of a path or cycle is its number of edges. For a given graph F, we let nF denote the graph with n components, all isomorphic to F. Given a family $\mathcal{F}$ of graphs, a graph G is *$\mathcal{F}$-free* if no induced subgraph of G is isomorphic to any member of $\mathcal{F}$; when $\mathcal{F}$ has only one element F we say that G is *F-free*. Whenever we say that a graph G contains a graph F, we mean that some induced subgraph of G is isomorphic to F.

In a graph G, a *k-hole* is an induced cycle on k vertices. A *hole* is any k-hole with $k \geq 4$. A *k-antihole* is the complementary graph of a k-hole. The neighborhood of a vertex x is denoted by $N_G(x)$ or $N(x)$ if there is no ambiguity. For any set $A \subseteq V(G)$ and vertex $x \in V(G)$, we let $N_A(x)$ denote the set $N_G(x) \cap A$. We say that a vertex x is *complete* to a set $S \subseteq V(G) \setminus \{x\}$ if x is adjacent to every vertex in S, and that x is *anticomplete* to S if x has no neighbor in S. Given disjoint sets $S, T \subseteq V(G)$, we say that S is *complete* to T if every vertex in S is adjacent to every vertex in T, and that S is *anticomplete* to T if no vertex in S has any neighbor in T. We let $\overline{G}$ denote the complementary graph of G.

We define three classes of graphs $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as follows (see Figure 1).

- **Class $\mathcal{A}$:** A graph G is in class $\mathcal{A}$ if $V(G)$ can be partitioned into two non-empty sets X and $\{a, b, c, d, e\}$ such that:
  - $\{a, b, c, d\}$ induces a hole with edges ab, bc, cd, da;
  - X induces a clique and is complete to $\{c, d\}$ and anticomplete to $\{a, b\}$;
  - e is complete to X, anticomplete to $\{a, b\}$, and has a non-neighbor in $\{c, d\}$.
- **Class $\mathcal{B}$:** A graph G is in class $\mathcal{B}$ if $V(G)$ can be partitioned into four stable sets X, Y, Z, W, with two special vertices $x \in X$ and $y \in Y$, such that:
  - $|X| \geq 2$, $|Y| \geq 2$, and $X \cup Y$ induces a connected P_5-free bipartite graph;
  - x is complete to Y, and y is complete to X;
  - Z is complete to $\{x, y\}$ and anticomplete to $(X \cup Y) \setminus \{x, y\}$;
  - W is anticomplete to $X \cup Y \cup Z$ (so all vertices of W are isolated in G).

  The structure of P_5-free bipartite graphs is recalled in Section 2.
- **Class $\mathcal{C}$:** A graph G is in class $\mathcal{C}$ if $V(G)$ can be partitioned in two cliques X and Y of size at least 2 such that the edges between X and Y form a matching of size 2.

A *split graph* [**4**] is any graph whose vertex-set can be partitioned into a clique and a stable set. Note that the complementary graph of a split graph is a split graph.

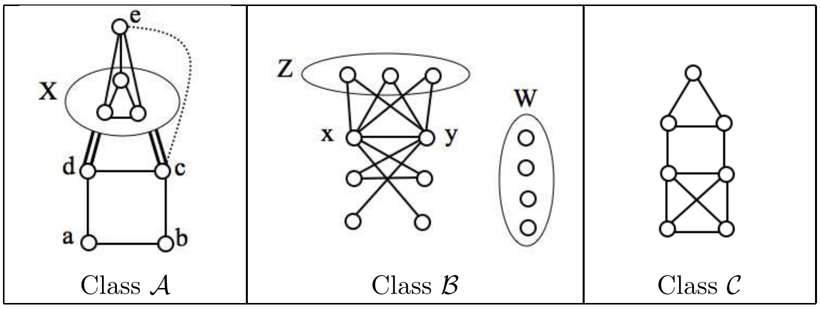

Figure 1.

Our main result is the following. Its proof is given in Section 3.

Theorem 1. *The following three properties are equivalent*:

(a) G *is* (*wheel, antiwheel*)*-free.*

(b) G *contains no wheel and no antiwheel on at most seven vertices.*

(c) G *or* $\overline{G}$ *is either a* 5*-hole, a* 6*-hole, a split graph, or a member of* $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.

2. P_5-FREE BIPARTITE GRAPHS AND SPLIT GRAPHS

We recall the following simple characterization of P_5-free bipartite graphs.

Theorem 2 (See [**5**] or [**8**, Section 2.4])**.** *Let* H *be a connected bipartite graph, where* $V(H)$ *is partitioned into stable sets* X *and* Y*. The following conditions are equivalent*:

- H *is* P_5*-free*;
- H *is* $2K_2$*-free*;
- *The neighborhoods of any two vertices in* X *are comparable by inclusion* (*equivalently, the same holds in* Y);
- *There is an integer* $h > 0$ *such that* X *can be partitioned into non-empty sets* $X_1, \ldots, X_h$ *and* Y *can be partitioned into non-empty sets* $Y_1, \ldots, Y_h$ *such that for all* $i, j \in \{1, \ldots, h\}$ *a vertex in* X_i *is adjacent to a vertex in* Y_j *if and only if* $i + j \leq h + 1$.

Using the properties described in this theorem one can also decide in linear time whether a bipartite graph is P_5-free [**5, 8**].

It follows from Theorem 2 that when H is a P_5-free connected bipartite graph, with the same notation as in the theorem, then X contains a vertex that is complete to Y (every vertex from X_1 has this property), and similarly Y contains a vertex that is complete to X (every vertex from Y_1 has this property).

Földes and Hammer [**4**] gave the following characterization of split graphs.

Theorem 3 ([**4**])**.** *A graph is split if and only if it is* $\{2K_2, C_4, C_5\}$*-free.*

3. THE PROOF

Proof of Theorem 1. Let F_1 (resp. F_2) be the wheel that consists of a 4-hole plus a vertex adjacent to three (resp. four) vertices of the hole.

Clearly, property (a) of Theorem 1 implies property (b).

Let us prove that (c) implies (a). Assume that G satisfies property (c). If G or $\overline{G}$ is a 5-hole or a 6-hole, then clearly it does not contain a wheel or an antiwheel. If G is a split graph (and so $\overline{G}$ too is a split graph), it contains no hole and consequently no wheel (and also no antiwheel). We may now assume that G or $\overline{G}$ is in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Actually we may assume that G is in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ since being (wheel, antiwheel)-free is a self-complementary property.

First we examine the presence of a wheel. If $G \in \mathcal{A} \cup \mathcal{C}$, it contains only one hole H, of length 4. If $G \in \mathcal{B}$ it may contain many holes, but they all have four vertices, more precisely two vertices from X and two from Y. In all cases, it is easy to see that whenever H is a hole in G, every vertex of $G \setminus H$ has at most two neighbors in H. So no hole of G extends to a wheel, and so G is wheel-free.

Now we examine the presence of an antiwheel. Note that G contains no 5-antihole (because in that case G is a 5-antihole, which we have already examined), and that in any k-antihole with $k \geq 6$ every vertex x has degree at least 3 and $N(x)$ is not a clique.

If $G \in \mathcal{A}$, it is easy to see that every antihole H of G has length 4 and consists of the vertices a and b plus two vertices u, v from $X \cup \{e\}$; moreover, c and d have three neighbors in H, while any vertex in $V(G) \setminus (V(H) \cup \{c, d\})$ is adjacent to both u, v; it follows that H cannot extend to an antiwheel ($\overline{F}_1$ or $\overline{F}_2$) in G.

If $G \in \mathcal{B}$, we claim that G contains no antihole at all. Indeed, G contains no 4-antihole ($= 2K_2$), by Theorem 2 and because there is no $2K_2$ containing a vertex from Z. Moreover, if H is a k-antihole in G with $k \geq 6$, then: clearly H contains no vertex from W; and H contains no vertex $z \in Z$ (because $N_G(z)$ is a clique); and so $V(H) \subseteq X \cup Y$, which is impossible because H must contain triangles. Thus the claim that G contains no antihole is established, and consequently G contains no antiwheel.

Finally, if $G \in \mathcal{C}$, it is easy to see that every antihole H in G has length 4 and that there is no vertex u in $V(G) \setminus V(H)$ such that $V(H) \cup \{u\}$ induces an antihole ($\overline{F}_1$ or $\overline{F}_2$), so G contains no antiwheel.

Finally let us prove that (b) implies (c). Let G be a graph that contains no wheel and no antiwheel on at most seven vertices.

First, suppose that G contains a 5-hole C. Note that $V(C)$ also induces a

5-hole in $\overline{G}$. If there is any vertex x in $V(G) \setminus V(C)$, then x has either at least three neighbors in C or three non-neighbors in C, and so $V(C) \cup \{x\}$ induces a wheel in G or in $\overline{G}$. Thus no such x exists, and G is a 5-hole.

Now suppose that G contains a 6-hole C, with vertices $c_1, \dots, c_6$ and edges $c_i c_{i+1}$, with subscripts modulo 6. Pick any x in $V(G) \setminus V(C)$. Vertex x has at most two neighbors in C, for otherwise $V(C) \cup \{x\}$ induces a wheel in G. It follows that, up to symmetry, $N(x) \cap V(C)$ is equal either to $\{c_1\}$, $\{c_1, c_2\}$, $\{c_1, c_5\}$, $\{c_1, c_4\}$ or $\emptyset$. In the first three cases $\{x, c_1, c_3, c_4, c_6\}$ induces an $\overline{F}_1$; in the last two cases $\{x, c_2, c_3, c_5, c_6\}$ induces an $\overline{F}_2$. Thus no such x exists, and G is a 6-hole.

If G contains a 6-antihole, then the same argument as in the preceding paragraph, applied to $\overline{G}$, implies that G is a 6-antihole.

We assume henceforth that G contains no 5-hole (and consequently no 5-antihole), no 6-hole and no 6-antihole. We may also assume that G is not a split graph, for otherwise the theorem holds. It follows from Theorem 3 that G contains either a $2K_2$, a C_4 or a C_5. Since G contains no C_5, and up to self-complementation, we may assume that G contains a $2K_2$. Let A, B be two disjoint subsets of $V(G)$ such that both A and B are cliques of size at least 2 and A is anticomplete to B. There exists such a pair since we can let A and B be the two cliques of size 2 of a $2K_2$. Choose A and B such that $|A \cup B|$ is maximized. Let $R = V(G) \setminus (A \cup B)$. We claim that:

(1) *For every vertex x in R, either*:
- *x is complete to A and has a neighbor in B, or*
- *x is complete to B and has a neighbor in A, or*
- *x has exactly one non-neighbor in A and exactly one non-neighbor in B.*

Suppose that the third item does not hold. So, up to symmetry, x has two non-neighbors a, a' in A. If x has a non-neighbor b in B, then, picking any $b' \in B \setminus b$, we see that $\{x, a, a', b, b'\}$ induces an $\overline{F}_1$ or $\overline{F}_2$ (depending on the pair x, b'), a contradiction. So x is complete to B. If x has no neighbor in A, then the pair $A, B \cup \{x\}$ contradicts the choice of A, B. So x has a neighbor in A, and the first item in (1) holds. This proves (1).

Let $A = \{a_1, \dots, a_p\}$, with $p \geq 2$, and let $B = \{b_1, \dots, b_q\}$, with $q \geq 2$. Define the following subsets of R:
- $R_0 = \{x \in R \mid x \text{ is complete to } A \text{ or to } B\}$.
- $R_{i,j} = \{x \in R \mid x \text{ is complete to } (A \cup B) \setminus \{a_i, b_j\} \text{ and anticomplete to } \{a_i, b_j\}\}$, for each $(i, j) \in \{1, \dots, p\} \times \{1, \dots, q\}$.

Clearly these sets are pairwise disjoint, and by (1) we have $R = R_0 \cup \bigcup_{i,j} R_{i,j}$.

Say that two vertices x and y of R are *A-comparable* if one of the two sets $N_A(x)$ and $N_A(y)$ contains the other; in the opposite case, say that x and y are *A-incomparable*. Define the same with respect to B.

Suppose that there are two A-incomparable vertices x and y in R. Up to relabeling, a_1 is adjacent to x and not to y and a_2 is adjacent to y and not to x. Since each of x and y has a neighbor in B, there is a path P between x and y with interior in B, and we may assume that P has no chord except possibly xy (if x, y

are adjacent). Since B is a clique, the length ℓ of P is equal to 2 or 3. We may assume that if $\ell = 2$ then $P = x\text{-}b_1\text{-}y$ while if $\ell = 3$ then $P = x\text{-}b_1\text{-}b_2\text{-}y$. Vertices x and y are adjacent, for otherwise $V(P) \cup \{a_1, a_2\}$ induces a 5-hole or a 6-hole.

(2) x and y are anticomplete to $A \setminus \{a_1, a_2\}$.

For suppose up to symmetry that x has a neighbor a in $A\setminus\{a_1, a_2\}$. Then $\{a_1, a_2, x, y, a\}$ induces an F_1 or F_2. Thus (2) holds.

(3) No vertex of R is complete to $\{a_1, a_2\}$.

Suppose that some z in R is complete to $\{a_1, a_2\}$. So $z \notin \{x, y\}$. Then z is anticomplete to $\{x, y\}$, for otherwise $\{x, y, z, a_1, a_2\}$ induces an F_1 or F_2. Then z is not adjacent to b_1, for otherwise either $\{x, y, z, b_1, a_1, a_2\}$ induces a 6-antihole (if $\ell = 2$) or $\{x, y, a_2, z, b_1\}$ induces a 5-hole (if $\ell = 3$). By (1) z has a neighbor b in B; so $b \neq b_1$. Then x is adjacent to b, for otherwise $\{x, a_1, z, b, b_1\}$ induces a 5-hole, and y is adjacent to b, for otherwise $\{x, y, a_2, z, b\}$ induces a 5-hole; but then $\{x, y, z, b, a_1, a_2\}$ induces a 6-antihole. Thus (3) holds.

Suppose that we can choose P with $\ell = 3$. Then $\{a_1, a_2\}$ and $\{b_1, b_2\}$ play symmetric roles. By (1), (3) and its analogue for $\{b_1, b_2\}$, we have $R = R_{1,1} \cup R_{1,2} \cup R_{2,1} \cup R_{2,2}$. Note that $x \in R_{2,2}$ and $y \in R_{1,1}$. If $p \geq 3$, then $\{x, y, a_1, a_2, a_3\}$ induces an F_2. So $p = 2$, and similarly $q = 2$. If there is any vertex u in $R_{1,2}$, then u is adjacent to x, for otherwise $\{u, b_1, x, a_1, a_2\}$ induces a 5-hole, and similarly u is adjacent to y; but then $\{u, x, y, a_1, a_2\}$ induces an F_1. So $R_{1,2} = \emptyset$, and similarly $R_{2,1} = \emptyset$. Therefore $V(G) = \{a_1, a_2, b_1, b_2\} \cup R_{1,1} \cup R_{2,2}$. If some vertex u in $R_{1,1}$ is not adjacent to some vertex v in $R_{2,2}$, then $\{u, a_1, a_2, v, b_2, b_1\}$ induces a 6-hole. So $R_{1,1}$ is complete to $R_{2,2}$. If $R_{1,1}$ contains two adjacent vertices u, v, then $\{u, v, x, a_1, a_2\}$ induces an F_1. So $R_{1,1}$ is a stable set, and similarly $R_{2,2}$ is a stable set. Thus $\overline{G}$ is in class $\mathcal{C}$ (where $R_{1,1} \cup \{a_1, b_1\}$ and $R_{2,2} \cup \{a_2, b_2\}$ are the two cliques that form a partition of $V(\overline{G})$ as in the definition of class $\mathcal{C}$).

Therefore we may assume that $\ell = 2$ and that there is no path P as above with $\ell = 3$, which means that x and y are B-comparable. We claim that:

(4) $R = \{x, y\}$.

For suppose that there is a vertex z in $R \setminus \{x, y\}$. Suppose that z is anticomplete to $\{a_1, a_2\}$. By (1), z is complete to B and has a neighbor a in $A \setminus \{a_1, a_2\}$. By (2), a is anticomplete to $\{x, y\}$. Then z is adjacent to x, for otherwise $\{x, a_1, a, z, b_1\}$ induces a 5-hole; and similarly z is adjacent to y. But then $\{x, y, z, a_1, a_2, a\}$ induces a 6-antihole. Therefore, by (3), z has exactly one neighbor in $\{a_1, a_2\}$. Up to symmetry, assume that z is adjacent to a_1 and not to a_2. If z is adjacent to b_1, then it is also adjacent to y, for otherwise $\{z, a_1, a_2, y, b_1\}$ induces a 5-hole, and to x, for otherwise $\{z, a_1, x, b_1, y\}$ induces an F_1; but then $\{x, y, a_1, a_2, z\}$ induces an F_1. So z is not adjacent to b_1, and so $z \in R_{2,1}$. Then z is adjacent to y, for otherwise either $\{z, a_1, a_2, y, b_1, b_2\}$ or $\{z, a_1, a_2, y, b_2\}$ induces a hole (depending on the adjacency between y and b_2), and z is not adjacent to x for otherwise $\{x, y, a_1, a_2, z\}$ induces an F_1. Then b_2 is adjacent to x, for otherwise $\{x, b_1, b_2, z, a_1\}$ induces a 5-hole, and to y, for otherwise $\{y, b_1, b_2, z, x\}$ induces an F_1. But then $\{a_1, z, b_2, x, y\}$ induces an F_1. Thus (4) holds.

If $p \geq 3$, then, by (2) and (1), x and y are anticomplete to $A \setminus \{a_1, a_2\}$ and complete to B. It follows that G is in class $\mathcal{C}$ (where the two cliques A and $B \cup \{x, y\}$ form a partition of $V(G)$ as in the definition of class $\mathcal{C}$). Now suppose that $p = 2$. Since x and y are B-comparable, we may assume, up to symmetry, that $N_B(x) \subseteq N_B(y)$. If B contains two vertices b, b' that are not adjacent to x, then $\{x, a_1, a_2, b, b'\}$ induces an $\overline{F}_1$. So B has at most one non-neighbor of x. If there is such a vertex b, then G is in class $\mathcal{A}$ (where $\{a_1, a_2, x, y\}$ induces a 4-hole, the set $B \setminus \{b\}$ plays the role of "X" and b plays the role of "e" in the definition of class $\mathcal{A}$). If there is no such vertex, then G is in class $\mathcal{C}$ (where $V(G)$ is partitioned into the two cliques A and $B \cup \{x, y\}$).

Therefore we may assume that any two vertices in R are A-comparable and B-comparable. By (1), every vertex of R has a neighbor in A, so some vertex of A is complete to R. Likewise, some vertex of B is complete to R. So we may assume that a_1 and b_1 are complete to R. If R is neither a clique nor a stable set, there are three vertices x, y, z in R that induce a subgraph with one or two edges, and then $\{a_1, b_1, x, y, z\}$ induces an F_1 or F_2, a contradiction. Therefore R is either a clique or a stable set.

Suppose that R is not a clique. So R is a stable set of size at least 2. For $\varepsilon \in \{0, 1\}$, let

$$A_\varepsilon = \{u \in A \setminus \{a_1\} \mid u \text{ has exactly } \varepsilon \text{ neighbors in } R\},$$
$$B_\varepsilon = \{u \in B \setminus \{b_1\} \mid u \text{ has exactly } \varepsilon \text{ neighbors in } R\}.$$

A vertex a in $A \setminus \{a_1\}$ cannot have two neighbors x and y in R, for otherwise $\{a, a_1, x, y, b_1\}$ induces an F_1. So $A = \{a_1\} \cup A_0 \cup A_1$. Likewise $B = \{b_1\} \cup B_0 \cup B_1$. Since any two vertices in R are A-comparable, some vertex x in R is complete to A_1, and $R \setminus \{x\}$ is anticomplete to $A \setminus \{a_1\}$. Likewise, some vertex y in R is complete to B_1, and $R \setminus \{y\}$ is anticomplete to $B \setminus \{b_1\}$. Suppose that $x = y$. Consider any $z \in R \setminus \{x\}$ (recall that $|R| \geq 2$). Then z is anticomplete to $(A \setminus \{a_1\}) \cup (B \setminus \{b_1\})$, so, by (1), we have $p = q = 2$. If x is anticomplete to $\{a_2, b_2\}$, then $\overline{G}$ is in class $\mathcal{C}$ (where $V(\overline{G})$ can be partitioned into two cliques $\{a_1, b_1\}$ and $R \cup \{a_2, b_2\}$). If x is not anticomplete to $\{a_2, b_2\}$, then $\overline{G}$ is in class $\mathcal{A}$ (where $\{a_1, b_1, a_2, b_2\}$ induces a 4-hole in $\overline{G}$, and $R \setminus \{x\}$ plays the role of the set "X", and x plays the role of the vertex "e"). Now suppose that we cannot choose x and y equal. So both A_1 and B_1 are not empty, and we may assume that a_2 is adjacent to x and not to y, and that b_2 is adjacent to y and not to x. If there is a vertex a_0 in A_0, then $\{a_0, a_2, x, y, b_2\}$ induces an $\overline{F}_1$. So $A_0 = \emptyset$. Likewise $B_0 = \emptyset$. If there is any vertex z in $R \setminus \{x, y\}$, then $\{x, y, z, a_2, b_2\}$ induces an $\overline{F}_2$. So $R = \{x, y\}$. Thus G is in class $\mathcal{C}$ (where $A \cup \{x\}$ and $B \cup \{y\}$ are two cliques that form a partition of $V(G)$).

Finally assume that R is a clique. Since any two vertices of R are A-comparable and B-comparable, there is at most one pair (i, j) such that $R_{i,j} \neq \emptyset$, and since a_1 and b_1 are complete to R, we may assume that if the pair (i, j) exists then $(i,j) = (2,2)$. Hence $R = R_0 \cup R_{2,2}$. Let

$$R^* = \{x \in R_0 \mid x \text{ is complete to } A \cup B\},$$
$$R_A = \{x \in R_0 \setminus R^* \mid x \text{ is complete to } A\},$$
$$R_B = \{x \in R_0 \setminus R^* \mid x \text{ is complete to } B\}.$$

So $R = R^* \cup R_A \cup R_B \cup R_{2,2}$, and $A \cup R_A$ and $B \cup R_B$ are cliques. Since any two vertices in R are A-comparable and B-comparable, the bipartite subgraph of $\overline{G}$ induced by $A \cup R_A \cup B \cup R_B$ is $2K_2$-free. By the definition of R_B and $R_{2,2}$, every vertex in $R_B \cup R_{2,2}$ has a non-neighbor in A, and since vertices in R are A-comparable, there is a vertex a in A that is anticomplete (in G) to $R_B \cup R_{2,2}$. Likewise there is a vertex b in B that is anticomplete in G to $R_A \cup R_{2,2}$. (If $R_{2,2} \neq \emptyset$, then $a = a_2$ and $b = b_2$.) By Theorem 2 it follows that $\overline{G}$ is in class $\mathcal{B}$ (where the four stable sets are $A \cup R_A$, $B \cup R_B$, $R_{2,2}$ and R^*, and a, b play the role of x, y). This completes the proof of the theorem. □

Property (b) of Theorem 1 implies that deciding whether a graph on n vertices and m edges is (wheel, antiwheel)-free can be done by brute force in time $O(n^7)$. So the problem is polynomially solvable. However, we can use property (c) of Theorem 1 to solve the problem in time $O(n+m)$, as follows:

- Test whether G is a 5-hole or a 6-hole. This can be done in time $O(n)$.
- Test whether G is a split graph. This can be done in time $O(n+m)$ as proved in [**6**].
- Test whether G or $\overline{G}$ is in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. This can be done in time $O(n+m)$ as explained in Theorem 4 below.

If any of the test fails, then G is not wheel-free or not antiwheel-free.

Theorem 4. *One can decide in time $O(m+n)$ whether a graph G on n vertices and m edges satisfies the property that either G or $\overline{G}$ is in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.*

Proof. Roughly, the algorithm will find vertices of certain degrees and from these vertices construct a partition of $V(G)$ as required in the definition of the classes. For all $i \in \{0, \dots, n-1\}$ let D_i be the set of vertices of degree i.

First we test whether $G \in \mathcal{A}$. Note that in a graph in $\mathcal{A}$ (with the same notation as in the definition of $\mathcal{A}$) the set of vertices of degree 2 is either $\{a, b\}$ or $\{a, b, e\}$, and in this second case, we have $|X| \in \{1, 2\}$ and $|V(G)| \in \{6, 7\}$. So we proceed as follows. Find the set D_2 of vertices of degree 2 in G. If either $|D_2| \notin \{2, 3\}$, or $|D_2| = 3$ and $|V(G)| \notin \{6, 7\}$, or $|D_2| = 2$ and the vertices in D_2 are not adjacent, then declare that G is not in $\mathcal{A}$. If $|D_2| = 3$ and $|V(G)| \in \{6, 7\}$, then use brute force. If $|D_2| = 2$ and its vertices a, b are adjacent, then let c be the unique vertex in $N(a) \setminus \{b\}$, let d be the unique vertex in $N(b) \setminus \{a\}$, and let $X = N(c) \cap N(d)$. Check that X is a clique, that there is a unique vertex e in $V(G) \setminus (\{a, b, c, d\} \cup X)$, and that e is complete to X and not complete to $\{c, d\}$.

Determining D_2, a, b, c, d, X, e and checking the properties can be done in time $O(m+n)$ by scanning the adjacency lists.

Testing whether $\overline{G} \in \mathcal{A}$ can be done similarly, starting from the set D_{n-3} of vertices of degree $n-3$ (instead of D_2), and arguing similarly, with adjacency and non-adjacency swapped. (It is not necessary to build the complementary graph $\overline{G}$.) So this can also be done in time $O(m+n)$ by scanning the adjacency lists.

Now we test whether $G \in \mathcal{B}$. We describe a graph in $\mathcal{B}$ with the same notation as in the definition of $\mathcal{B}$ and, for the bipartite graph induced by $X \cup Y$, with the same notation (the sets $X_1, \ldots, X_h$, $Y_1, \ldots, Y_h$) as in Theorem 2. Note that if $h = 1$, then x and y are universal vertices in $G \setminus W$. If $h \geq 2$, then $G \setminus W$ has no universal vertex but it has vertices of degree 1 (at least one in X_h and one in Y_h, actually $X_h \cup Y_h = D_1$), and they form a stable set, and they are all adjacent to either x or y. So we proceed as follows. Determine the set W $(= D_0)$ of isolated vertices in G. Determine the set U of universal vertices of $G \setminus W$ (so $U = D_{n-1-|W|}$). If $|U| \geq 2$, pick any two vertices $x, y \in U$; then if $V(G) \setminus (W \cup \{x, y\})$ is a stable set, declare that $G \in \mathcal{B}$, else declare that $G \notin \mathcal{B}$. If $|U| = 1$, declare that $G \notin \mathcal{B}$. Now suppose that $U = \emptyset$. Let D_1 be the set of vertices of degree 1. If either $|D_1| \leq 1$, or D_1 is not a stable set, or $N(D_1)$ does not consist of two adjacent vertices, declare that $G \notin \mathcal{B}$. Now suppose that $|D_1| \geq 2$, D_1 is a stable set, and $N(D_1)$ consists of two adjacent vertices x, y. Let $Z = N(x) \cap N(y)$, and $X = N(y) \setminus Z$ and $Y = N(x) \setminus Z$. Check whether $V(G) \setminus (W \cup Z \cup X \cup Y) = \emptyset$. Check whether X and Y are stable sets and whether $X \cup Y$ induces a P_5-free bipartite graph (as explained after Theorem 2). Check whether Z is a stable set and is anticomplete to $V(G) \setminus \{x, y\}$. Determining D_1, x, y, Z, X, Y and checking the properties can be done in time $O(m+n)$ by scanning the adjacency lists.

Testing whether $\overline{G} \in \mathcal{B}$ can be done similarly, starting from the set $W' = D_{n-1}$ of universal vertices (instead of W), the set $U' = D_{|W'|}$ of isolated vertices in $G \setminus W'$ (instead of U), and the set D_{n-2} of vertices that have exactly one non-neighbor (instead of D_1), and arguing similarly, with adjacency and non-adjacency swapped.

Finally we test whether $G \in \mathcal{C}$. We describe a graph in $\mathcal{C}$ with the same notation as in the definition of $\mathcal{C}$, assuming witout loss of generality that $|Y| \leq |X|$. If $|Y| = 2$, then the graph either has at most five vertices (if $|X| \leq 3$) or has the same structure as a graph in class $\mathcal{A}$ minus the vertex e (where the two vertices in Y play the role of a, b); this can be tested with a variant of the algorithm for class $\mathcal{A}$ (just forgetting the instructions that deal with vertex e). Now suppose that $|Y| \geq 3$. Then there is a vertex in Y with no neighbor in X, and any such vertex has minimum degree in G, and every vertex of minimum degree in G is such a vertex (or is a vertex in X with no neighbor in Y, in case $|X| = |Y|$). So we proceed as follows. Let y be a vertex of minimum degree in G. Let $Y = \{y\} \cup N(y)$ and $X = V(G) \setminus Y$. Check that X and Y are cliques, and that there are exactly two, non-incident, edges between them. Determining y, X, Y and checking the properties can be done in time $O(m+n)$ by scanning the adjacency lists.

Testing whether $\overline{G} \in \mathcal{C}$ can be done similarly, starting from a vertex y of maximum degree (instead of minimum) and arguing similarly, with adjacency and non-adjacency swapped. This completes the proof.

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